International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2008**

1 [4] A square board is divided by lines parallel to the board sides (7 lines in each direction, not necessarily equidistant) into 64 rectangles. Rectangles are colored into white and black in alternating order. Assume that for any pair of white and black rectangles the ratio between area of white rectangle and area of black rectangle does not exceed 2. Determine the maximal ratio between area of white and black part of the board. White (black) part of the board is the total sum of area of all white (black) rectangles.

2 [6] Space is dissected into congruent cubes. Is it necessarily true that for each cube there exists another cube so that both cubes have a whole face in common?

3 [6] There are N piles each consisting of a single nut. Two players in turns play the following game. At each move, a player combines two piles that contain coprime numbers of nuts into a new pile. A player who can not make a move, loses. For every $N > 2$ define which of the players, the first or the second has a winning strategy.

4 [6] Let $ABCD$ be a non-isosceles trapezoid. Define a point A_1 as intersection of circumcircle of triangle BCD and line AC. (Choose A_1 distinct from C). Points B_1, C_1, D_1 are defined in similar way. Prove that $A_1B_1C_1D_1$ is a trapezoid as well.

5 [8] In an infinite sequence $a_1, a_2, a_3, \ldots$, the number a_1 equals 1, and each $a_n,\ n > 1$, is obtained from a_{n-1} as follows:

- if the greatest odd divisor of n has residue 1 modulo 4, then $a_n = a_{n-1} + 1$,
- and if this residue equals 3, then $a_n = a_{n-1} - 1$.

Prove that in this sequence each positive integer occurs infinitely many times.
(The initial terms of this sequence are 1, 2, 1, 2, 3, 2, 1, 2, 3, 4, 3, ...)

6 [9] Let $P(x)$ be a polynomial with real coefficients so that equation $P(m) + P(n) = 0$ has infinitely many pairs of integer solutions (m, n). Prove that graph of $y = P(x)$ has a center of symmetry.

7 A test consists of 30 true or false questions. After the test (answering all 30 questions), Victor gets his score: the number of correct answers. Victor is allowed to take the test (the same questions) several times. Can Victor work out a strategy that insure him to get a perfect score after

(a) [5] 30th attempt?

(b) [5] 25th attempt?

(Initially, Victor does not know any answer)