International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper[1] **Spring 2007.**

1. A, B, C and D are points on the parabola $y = x^2$ such that AB and CD intersect on the y-axis. Determine the x-coordinate of D in terms of the x-coordinates of A, B and C, which are a, b and c respectively.

2. A convex figure F is such that any equilateral triangle with side 1 has a parallel translation that takes all its vertices to the boundary of F. Is F necessarily a circle?

3. Let $f(x)$ be a polynomial of nonzero degree. Can it happen that for any real number a, an even number of real numbers satisfy the equation $f(x) = a$?

4. Nancy shuffles a deck of 52 cards and spreads the cards out in a circle face up, leaving one spot empty. Andy, who is in another room and does not see the cards, names a card. If this card is adjacent to the empty spot, Nancy moves the card to the empty spot, without telling Andy; otherwise nothing happens. Then Andy names another card and so on, as many times as he likes, until he says "stop."
   (a) Can Andy guarantee that after he says "stop," no card is in its initial spot?
   (b) Can Andy guarantee that after he says "stop," the Queen of Spades is not adjacent to the empty spot?

5. From a regular octahedron with edge 1, cut off a pyramid about each vertex. The base of each pyramid is a square with edge $\frac{1}{3}$. Can copies of the polyhedron so obtained, whose faces are either regular hexagons or squares, be used to tile space?

6. Let a_0 be an irrational number such that $0 < a_0 < \frac{1}{2}$. Define $a_n = \min\{2a_{n-1}, 1 - 2a_{n-1}\}$ for $n \geq 1$.
   (a) Prove that $a_n < \frac{3}{16}$ for some n.
   (b) Can it happen that $a_n > \frac{7}{40}$ for all n?

7. T is a point on the plane of triangle ABC such that $\angle ATB = \angle BTC = \angle CTA = 120°$. Prove that the lines symmetric to AT, BT and CT with respect to BC, CA and AB, respectively, are concurrent.

Note: The problems are worth 3, 5, 5, 4+4, 8, 4+4 and 8 points respectively.

[1]Courtesy of Professor Andy Liu