International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper[1] **Spring 2007.**

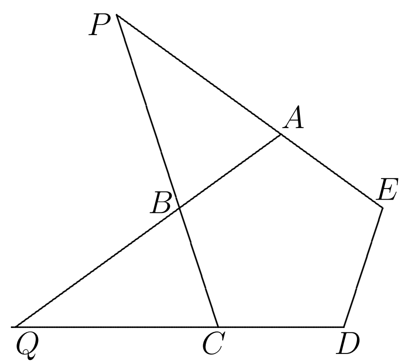
1. The sides of a convex pentagon are extended on both sides to form five triangles. If these triangles are congruent to one another, does it follow that the pentagon is regular?
2. Two 2007-digit numbers are given. It is possible to delete 7 digits from each of them to obtain the same 2000-digit number. Prove that it is also possible to insert 7 digits into the given numbers so as to obtain the same 2014-digit number.
3. What is the least number of rooks that can be placed on a standard 8×8 chessboard so that all the white squares are attacked? (A rook also attacks the square it is on, in addition to every other square in the same row or column.)
4. Three nonzero real numbers are given. If they are written in any order as coefficients of a quadratic trinomial, then each of these trinomials has a real root. Does it follow that each of these trinomials has a positive root?
5. A triangular pie has the same shape as its box, except that they are mirror images of each other. We wish to cut the pie in two pieces which can fit together in the box without turning either piece over. How can this be done if
   (a) one angle of the triangle is three times as big as another;
   (b) one angle of the triangle is obtuse and is twice as big as one of the acute angles?

Note: The problems are worth 4, 4, 4, 4 and 1+4 points respectively.

[1] Courtesy of Professor Andy Liu

Solution to Junior O-Level Spring 2007

1. Let $ABCDE$ be the pentagon and PAB and QBC be two of the triangles formed. Let $\angle PAB = \alpha$, $\angle ABP = \beta$ and $\angle BPA = \gamma$. Then $\angle QBC = \beta$ also. Since the two triangles are congruent to each other, $\angle CQB = \alpha$ or γ. It cannot be α as otherwise AE and CD would be parallel, and their extensions will not meet each other to form one of the five triangles. Hence we must have $\angle CQB = \gamma$, and it follows that all five angles away from the pentagon are equal to γ. This means that the angles of the pentagon are equal to $180° - \alpha$ and $180° - \beta$ alternately. Since five is an odd number, this means that $\alpha = \beta$, and the pentagon is equiangular. It is also equilateral as its sides are the sides of the triangles facing the angles equal to γ. Hence the pentagon must be regular.

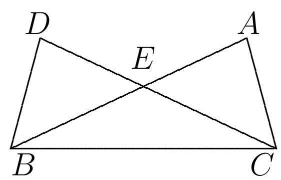


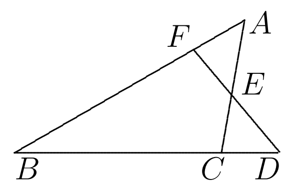
2. To the 2000-digit number, add back the 7 digits deleted from the two 2007-digit numbers. Digits inserted between the same two digits of the 2000-digit number can be arranged in any order. This yields a 2014-digit number. It is obtainable from either 2007-digit number by inserting the 7 digits from the other 2007-digit number.

3. A rook on a black square attacks exactly 8 white squares. A rook on a white square attacks exactly 7 white squares. Since there are 32 white squares overall, we need at least 4 rooks. Labelling the ranks a to h and the files 1 to 8 and using the convention that a1 is a black square, we can show that 4 is indeed the minimum by placing a rook on each of a7, c5, e3 and g1.

4. Since the three numbers a, b and c are all non-zero, 0 is not a root of any of the six trinomials under consideration. Suppose that $ax^2 + bx + c$ has two negative roots $-r$ and $-s$, where r and s are positive numbers. Then $ax^2 + bx + c = a(x + r)(x + s)$, so that $b = a(r + s)$ and $c = ars$ both have the same sign as a. Hence we may assume that they are all positive. Since one of the roots is real, both are real, so that we have $b^2 \geq 4ac$. Similarly, $c^2 \geq 4ab$ and $a^2 \geq 4bc$. Multiplication yields $(abc)^2 \geq (8abc)^2$, which is a contradiction. It follows that each of the six trinomials has a positive root.

5. (a) Let the pie be represented by triangle ABC in which $\angle ABC = \theta$ and $\angle BCA = 3\theta$. Let D be the image of A under the reflection about the perpendicular bisector of BC. (See the diagram below to the left.) Let AB and DC intersect at E. Then triangle DBC represents the box. Now $\angle BCD = \angle ABC = \theta$ so that $\angle DCA = \angle BCA - \angle BCD = 2\theta$. On the other hand, $\angle AEC = \angle ABC + \angle BCD = 2\theta$ also. Hence triangle AEC is isosceles, as is triangle DEB which is congruent to it by reflection. Hence if we cut the pie along CE, we can fit the two pieces inside the box.





(b) Let the pie be represented by triangle ABC in which $\angle CAB = \theta$ and $\angle BCA = 2\theta$. Let D and F be the respective images of A and C under the reflection about the bisector of $\angle ABC$. (See the diagram above to the right.) Let AC and DF intersect at E. Then triangle DBF represents the box. Now $\angle BDF = \angle CAB = \theta$. It follows that

$$\angle DEC = \angle BCA - \angle BDF = \theta = \angle BDF.$$

Hence triangle CDE is isosceles, as is triangle FAE which is congruent to it by reflection. It follows that if we cut the pie along EF, we can fit the two pieces inside the box.