International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper[1] **Spring 2007.**

1. Let n be a positive integer. In order to find the integer closest to $\sqrt{n}$, Mary finds a^2, the closest perfect square to n. She thinks that a is then the number she is looking for. Is she always correct?

2. K, L, M and N are points on sides AB, BC, CD and DA, respectively, of the unit square $ABCD$ such that KM is parallel to BC and LN is parallel to AB. The perimeter of triangle KLB is equal to 1. What is the area of triangle MND?

3. Anna's number is obtained by writing down 20 consecutive positive integers, one after another in arbitrary order. Bob's number is obtained in the same way, but with 21 consecutive positive integers. Can they obtain the same number?

4. Several diagonals (possibly intersecting each other) are drawn in a convex n-gon in such a way that no three diagonals intersect in one point. If the n-gon is cut into triangles, what is the maximum possible number of these triangles?

5. Find all (finite) increasing arithmetic progressions, consisting only of prime numbers, such that the number of terms is larger than the common difference.

6. In the quadrilateral $ABCD$, $AB = BC = CD$ and $\angle BMC = 90°$, where M is the midpoint of AD. Determine the acute angle between the lines AC and BD.

7. Nancy shuffles a deck of 52 cards and spreads the cards out in a circle face up, leaving one spot empty. Andy, who is in another room and does not see the cards, names a card. If this card is adjacent to the empty spot, Nancy moves the card to the empty spot, without telling Andy; otherwise nothing happens. Then Andy names another card and so on, as many times as he likes, until he says "stop."

   (a) Can Andy guarantee that after he says "stop," no card is in its initial spot?

   (b) Can Andy guarantee that after he says "stop," the Queen of Spades is not adjacent to the empty spot?

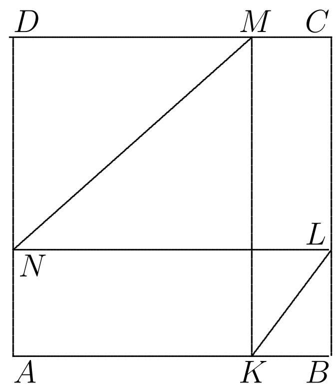
Note: The problems are worth 3, 4, 5, 6, 7, 8 and 5+5 points respectively.

[1] Courtesy of Professor Andy Liu

Solution to Junior A-Level Spring 2007

1. If $n = a^2$, clearly a is the right choice for $/sqrtn$. Suppose $a^2 < n < (a+1)^2$. Then Mary would choose a or $a+1$ according to whether $a^2 < n \leq a^2 + a$ or $a^2 + a + 1 \leq n < a^2 + 2n + 1$. In the former case, we have $a^2 < n < a^2 + a + \frac{1}{2}$ so that $a < \sqrt{n} < a + \frac{1}{2}$. In the latter case, we have $a + \frac{1}{2} < \sqrt{n} < a + 1$. So Mary is always correct.

2. Let $AK = x$ and $AN = y$. From $1 - (1-x) - y = 1 - KB - LB = KL = \sqrt{(1-x)^2 + y^2}$, we have $x^2 - 2xy + y^2 = 1 - 2x + x^2 + y^2$, which simplifies to $2x(1-y) = 1$. Hence the area of triangle DMN is equal to $\frac{1}{2}DM \cdot DN = \frac{1}{2}x(1-y) = \frac{1}{4}$.



3. Anna's number can be 4567891011121314151617181920212223, obtained by writing down the 20 integers from 4 to 23 inclusive, in their natural order. Bob can obtain the same number by writing down the 21 integers from 2 to 22 inclusive, in their natural order except that 2 and 3 are moved from the front to the back.

4. There are four kinds of triangles inside the n-gon. They have respectively 3, 2, 1 and 0 vertices which are vertices of the n-gon. We claim that only the first two kinds can exist. Suppose ABC is a triangle of the third kind, with A a vertex of the n-gon. Then B and C are the points of intersection of a diagonal not passing through A with two diagonals passing through A. However, the region separated from ABC by BC cannot be a triangle, since no other lines pass through B or C. This contradiction shows that a triangle of the third kind cannot exist. Similarly, we can show that neither can a triangle of the fourth kind. Triangles of the first kind stand on their own while triangles of the second kind come in groups of four, forming a convex quadrilateral. If we have two adjacent triangles of the first kind, we can add a diagonal and get four triangles of the second kind instead. If we have two triangles of the first kind separated by quadrilaterals, we can have one triangle trade places with the intervening quadrilaterals one at a time, until the two triangles are adjacent. This way, we are left with at most one triangle of the first kind if n is odd, and no triangles of the first kind if n is even. In summary, for $n = 2k$, we have $4k - 4$ triangles. For $n = 2k + 1$, we have $4k - 3$ triangles.

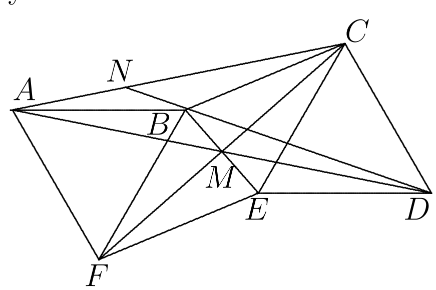
5. Let d be the common difference. Suppose there is a prime p less than d which does not divide d. Then every p-term of the progression is divisible by p. Since every term is a prime, the term divisible by p must be p itself, and it is therefore the first term of the progression. However, the $(p+1)$-st term will be a composite number which is divisible by p. Hence the maximum number of terms is $p < d$. It follows that in order for the number of terms to exceed d, we must have either $d = 1$ or d being the product of consecutive primes starting from 2.

For $d = 1$, every other term is divisible by 2. Hence the only such progression is $\{2, 3\}$. For $d = 2$, every third term is divisible by 3. Hence the only such progression is $\{3, 5, 7\}$. For $d = 2 \times 3$, the prime 5 will cause problem. For $d = 2 \times 3 \times 5$, the prime 7 will cause problem. In general, let p_i denotes the i-th prime. We claim that if $d = p_1 p_2 \cdots p_n$ for $n \geq 3$, then $d > p_{n+1}$. Define $m = p_2 p_3 \cdots p_n - 2$. Since $n \geq 3$, $m > 1$ and m has a prime divisor q. We cannot have $q = p_i$ for any $2 \leq i \leq n$, nor can we have $q = p_1 = 2$ since m is odd. Hence $q \geq p_{n+1}$ and it will cause problem. In summary, there are only two such progressions, namely, $\{2, 3\}$ and $\{3, 5, 7\}$.

6. Extend BM to E and CM to F so that $BM = ME$ and $CM = MF$. Then $ABCDEF$ is an equilateral hexagon. Since $\angle BMC = 90°$, $BCEF$ is a kite so that $BC = CE = EF = FB$. It follows that FAB and CDE are both equilateral triangles. Let AC intersect BD at N. In the diagram below, we have

$$\begin{aligned}
\angle CND &= \angle CBD - \angle BCA \\
&= \frac{1}{2}(180° - \angle BCD) - \frac{1}{2}(180° - \angle ABC) \\
&= \frac{1}{2}(120° - \angle BCE) - \frac{1}{2}(180° - (360° - \angle FBC - \angle ABF)) \\
&= \frac{1}{2}(120° + 180° - 60° - (\angle BCE + \angle FBC)) \\
&= 30°
\end{aligned}$$

since $\angle BCE + \angle FBC = 180°$. Note that the diagram may be different if $ABCD$ is convex, but the argument is essentially the same.



7. (a) The answer is yes. Andy can call the cards out in order starting with the Ace of Spades, two of Spades down to the King of Spades, followed by the Hearts, the Diamonds and the Clubs. We refer to this as one cycle. In each cycle, each card can move at most once since it is called exactly once, and at least one card must move. Andy then makes another 51 cycles of calls. We claim that all moves are in the same direction, either all clockwise or all counter-clockwise. This is clear within each cycle. Consider the card X which is the last to move in a cycle, and let Y be the other card adjacent to the empty spot. Since Y does not move after X in this cycle, it must have been called before X. So in the next cycle, Y will be called before X, and follows X in the same direction. This justifies our claim. To go once around and return to its initial spot, a card must have moved 53 times, and this is not possible since Andy makes only 52 cycles of calls. If it is to be in its initial spot, it must not have moved at all. However, this is also impossible as otherwise at most 1 move could have been made, but in 52 cycles, at least 52 moves have been made. Therefore, after 52 cycles of calls, every card is in a spot different from its initial one.

(b) The answer is no. Construct a graph where each of the vertices represents one of the 52! permutations of the cards, with the first and the last adjacent to the empty spot. Two vertices are joined by an edge if and only if a call by Andy changes the two permutations to each other. Label the edge with the card called by Andy. In this graph, each vertex has degree 2, and the graph is a union of disjoint cycles. Consider the cycle containing the vertex representing the initial permutation. For each vertex, let a person starts there. Whenever Andy makes a call, the person moves along an edge labelled with that card to an adjacent vertex if possible, and stays put otherwise. We call a vertex safe if and only if in the permutation it represents, the Queen of Spades is not adjacent to the empty spot. By shifting each card clockwise into the empty spot in turns, we will arrive at permutations represented by safe vertices as well as permutations represented by unsafe vertices. Note that after each call, there is still one person on each vertex. Thus no matter what sequence of calls Andy may employ, he cannot get everyone to a safe vertex. It follows that there is an initial permutation for which Andy's sequence will leave the Queen of Spades adjacent to the empty spot.