International Mathematics

TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2007.**

1. Pictures are taken of 100 adults and 100 children, with one adult and one child in each, the adult being the taller of the two. Each picture is reduced to $\frac{1}{k}$ of its original size, where k is a positive integer which may vary from picture to picture. Prove that it is possible to have the reduced image of each adult taller than the reduced image of every child.

2. Initially, the number 1 and two positive numbers x and y are written on a blackboard. In each step, we can choose two numbers on the blackboard, not necessarily different, and write their sum or their difference on the blackboard. We can also choose a non-zero number of the blackboard and write its reciprocal on the blackboard. Is it possible to write on the blackboard, in a finite number of moves, the number

   (a) x^2;

   (b) xy?

3. Give a construction by straight-edge and compass of a point C on a line ℓ parallel to a segment AB, such that the product $AC \cdot BC$ is minimum.

4. The audience chooses two of twenty-nine cards, numbered from 1 to 29 respectively. The assistant of a magician chooses two of the remaining twenty-seven cards, and asks a member of the audience to take them to the magician, who is in another room. The two cards are presented to the magician in an arbitrary order. By an arrangement with the assistant beforehand, the magician is able to deduce which two cards the audience has chosen only from the two cards he receives. Explain how this may be done.

5. A square of side length 1 centimetre is cut into three convex polygons. Is it possible that the diameter of each of them does not exceed

   (a) 1 centimetre;

   (b) 1.01 centimetres;

   (c) 1.001 centimetres?

Note: The problems are worth 3, 2+2, 4, 4 and 1+2+2 points respectively.

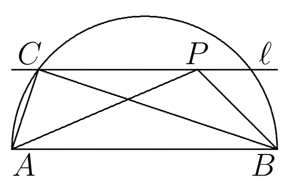
Solution to Senior O-Level Fall 2007

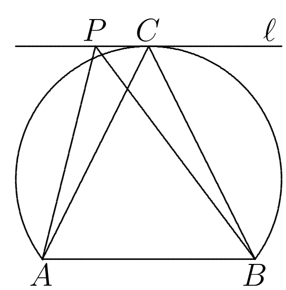
1. We use mathematical induction on the number n of pictures. For $n = 1$, there is nothing to do. Suppose the result holds for some $n \geq 1$. Consider the next case with $n + 1$ pictures. By the induction hypothesis, the first n photos can be reduced so the reduced image of each adult is taller than the reduced image of every child. Let the minimum reduced height of the adults be a and the maximum reduced height of the children be c. Let the height of the $(n + 1)$st adult be b and the height of the $(n + 1)$st child be d. If $b > c$ and $a > d$, there is nothing to prove. Suppose $b > d > a > c$. Then $\frac{c}{b} < \frac{d}{a}$. Hence there exists a rational number r such that $\frac{c}{b} < r < \frac{d}{a}$. It follows that $a > rd$ and $rb > c$. Let h and k be positive integers such that $r = \frac{h}{k}$. Then we reduce the $(n + 1)$st picture to $\frac{1}{h}$ of its original size and each of the other pictures further to $\frac{1}{k}$ of its reduced size. The case $a > c > b > d$ can be handled in exactly the same way.

2. (a) We first write down $x + 1$ and $x - 1$. Then we can write down $\frac{1}{x+1}$, $\frac{1}{x-1}$ and their sum $\frac{2}{x^2-1}$. The reciprocal of this number is $\frac{x^2-1}{2}$. Adding this number to itself yields $x^2 - 1$, and adding 1 to it yields x^2.

   (b) We first write down $x+y$. By (a), we can write down $(x+y)^2$, x^2 and y^2. This is followed by $2xy = (x + y)^2 - x^2 - y^2$ and $\frac{1}{2xy}$. Finally, we can write down the sum $\frac{1}{2xy} + \frac{1}{2xy} = \frac{1}{xy}$ and the reciprocal xy.

3. Draw a semicircle with diameter AB towards ℓ. Suppose they have a point of intersection C. Then $\angle BCA = 90°$ so that the area of triangle ABC is $\frac{1}{2}AC \cdot BC$. For any other point P on ℓ, the area of triangle ABP is $\frac{1}{2}AP \cdot BP \sin APB \leq \frac{1}{2}AP \cdot BP$. Since ℓ is parallel to AB, triangles ABC and ABP have the same area. It follows that $AC \cdot BC \leq AP \cdot BP$. Suppose the semicircle does not intersect ℓ. Let C be the point on ℓ equidistant from A and B. Draw the circumcircle of triangle ABC. Then ℓ is one of its tangents. Hence for any point P on ℓ other than C, $\angle ACB > \angle APB$. Since triangles ABC and ABP have the same area, we have $AC \cdot BC < AP \cdot BP$ as before.





4. The magician and her assistant can agree beforehand to arrange the numbers 1 to 29 in order on a circle, so that 1 follows 29. If the audience chooses two adjacent cards, say 3 and 4, the assistant will choose the two cards after them, which are 5 and 6. If the audience chooses two non-adjacent cards, say 3 and 29, the assistant will choose the cards after them, namely, 4 and 1. If the magician receives two adjacent cards, say 2 and 3, she will know that the audience must have chosen 29 and 1. If the magician receives two non-adjacent cards, say 2 and 15, she will know that the audience must have chosen 1 and 14.

5. (a) Let $ABCD$ be the square. Let E, F and G be the respective midpoints of AB, BC and CD. Let H be a point on GD at a distance $\frac{1}{8}$ from D. (See the diagram below on the left.) Suppose $ABCD$ has been dissected into three convex polygons each of diameter at most 1. By the Pigeonhole Principle, two of the vertices of the square must belong to the same polygon. They cannot be opposite vertices as otherwise the diameter of that polygon will be $\sqrt{2}$. Hence we may assume that A and D belong to the first polygon. Suppose B and C belong to the second. Since E is too far from C and from D, it must belong to the third polygon. However, H is too far from each of A, E and B, and cannot belong to any of the three polygons. From this contradiction, we may assume that B belongs to the second polygon while C belongs to the third. Then E must belong to the second polygon while G must belong to the third. Since F is too far from A, we may assume that it belongs to the third polygon. However, H is too far from A, from B and also from F since $HF^2 = (\frac{1}{2})^2 + (\frac{7}{8})^2 = \frac{65}{64}$. This contradiction shows that the task is impossible.

(b) We can cut a square $ABCD$ into three rectangles $ADHK$, $BFJK$ and $CFJH$, as shown in the diagram below on the right. The diameter of any rectangle is a diagonal. Now $AH^2 = 1^2 + (\frac{1}{8})^2 = \frac{65}{64}$. From (a), $FK^2 = FH^2 = \frac{65}{64}$ also. Now $\frac{1}{64} < \frac{1}{50} < \frac{201}{10000}$ so that $\frac{65}{64} < \frac{10201}{10000}$. It follows that the diameter of each of the rectangles is less than $\frac{101}{100}$.

(c) The task is impossible, and the argument is exactly the same as that in (a) since we have $AH = FH = \frac{\sqrt{65}}{8} > \frac{1001}{1000}$.