International Mathematics

TOURNAMENT OF THE TOWNS

Junior P-Level Paper **Fall 2007**

1 [**1**] (from The Good Soldier Švejk) Senior military doctor Bautze exposed *abccc* malingerers among *aabbb* draftees who claimed not to be fit for the military service. He managed to expose all but one draftees. (He would for sure expose this one too if the lucky guy was not taken by a stroke at the very moment when the doctor yelled at him "Turn around !…") Now many malingerers were exposed by the vigilant doctor?

Each digit substitutes a letter. The same digits substitute the same letters, while distinct digits substitute distinct letters.

Solution Problem is equivalent to: $aabbb = abccc + 1$,

Then, $c = 9$ (otherwise, nothing is carried out to the second digit); therefore, $b = 0$ and $a = 1$.

2 [**2**] Let us call a triangle "almost right angle triangle" if one of its angles differs from 90° by no more than 15°. Let us call a triangle "almost isosceles triangle" if two of its angles differs from each other by no more than 15°. Is it true that that any acute triangle is either "almost right angle triangle" or "almost isosceles triangle"?

Answer: Yes, it is true.

Solution. Let $a \geq b \geq c$ be angles of a triangle. Let us assume that a triangle in not "almost isosceles". Then $a - b > 15°$ and $b - c > 15°$ (so $a - c > 30°$). Then $180° = a + b + c < a + a + 15° + a + 30°$ or $3a > 225°$; so $a > 75°$. That implies that the triangle is "almost right angle triangle".

3 [**2**] A triangle with sides a, b, c is folded along a line ℓ so that a vertex C is on side c. Find the segments on which point C divides c, given that the angles adjacent to ℓ are equal.

Solution. Let ABC be a given triangle. It is clear that the folding along line is equivalent to the mirror reflection with respect to this line. Let point C' (on side AB) be an image of vertex C under mirror reflection with respect to line ℓ; thus, CC' is perpendicular to ℓ. Let M and N be points of intersection of ℓ with sides AC and CB respectively. Since angles adjacent to ℓ are equal then $\angle CMN = \angle CNM$ and triangle CMN is isosceles. Therefore, line CC' is an altitude of isosceles triangle. Then, CC' is also a bisector of $\angle C$. By a property of bisector we have $AC'/C'B = AC/CB$ or $AC' - (c - AC') = b/a$ and we get $C'B = ac/(a + b)$.

4 [**3**] From the first 64 positive integers are chosen two subsets with 16 numbers in each. The first subset contains only odd numbers while the second one contains only even numbers. Total sums of both subsets are the same. Prove that among all the chosen numbers there are two whose sum equals 65.

Solution. Let us pair the first 64 positive integers: $(i, 65 - i)$. It is easy to see that we have one-to-one correspondence between all odd and all even numbers of $\{1, \ldots, 64\}$. Let us pick up any $F \subset \{1, 3, \ldots, 63\}$ consisting of 16 numbers. Let us also pick up any $S \subset \{2, 4, \ldots, 64\}$

consisting of 16 numbers. If we do not want any element of S be paired with some element of F, then it it easy to see that S is uniquely defined by choice of F.

So, let $F = \{f_1, \ldots, f_{16}\}$ and $S = \{s_1, \ldots, s_{16}\}$. It is clear that $s_j \subset \ S$ if and only if $65 - s_j \notin F$. Therefore, to get the sum $\bar{s} = s_1 + \cdots + s_{16}$ we need to sum up $65 - k_j$ (k_j odd and $k_j \notin F$). Thus, we get $65 \cdot 16 - \sigma + \bar{f}$ where $\sigma = 1 + 3 + 5 + \cdots + 63 = 32^2$ and $\bar{f} = f_1 + f_2 + \cdots + f_{16}$. So, $\bar{s} = 65 \cdot 16 - 32^2 + \bar{f} = 16 + \bar{f} \neq \bar{f}$. Contradiction.

5 [**4**] Two players in turns color the squares of a 4×4 grid, one square at the time. Player loses if after his move a square of 2×2 is colored completely. Which of the players has the winning strategy, First or Second?

Solution. Second Player has a strategy. On each move of First Player, Second Player corresponding move is two squares down (or two squares up) in the same column. It is easy to see that if First Player has a move, so does Second Player.