International Mathematics

TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2007.**

1. Let $ABCD$ be a rhombus. Let K be a point on the line CD, other than C or D, such that $AD = BK$. Let P be the point of intersection of BD with the perpendicular bisector of BC. Prove that A, K and P are collinear.
2. (a) Each of Peter and Basil thinks of three positive integers. For each pair of his numbers, Peter writes down the greatest common divisor of the two numbers. For each pair of his numbers, Basil writes down the least common multiple of the two numbers. If both Peter and Basil write down the same three numbers, prove that these three numbers are equal to each other.
   (b) Can the analogous result be proved if each of Peter and Basil thinks of four positive integers instead?
3. Michael is at the centre of a circle of radius 100 metres. Each minute, he will announce the direction in which he will be moving. Catherine can leave it as is, or change it to the opposite direction. Then Michael moves exactly 1 metre in the direction determined by Catherine. Does Michael have a strategy which guarantees that he can get out of the circle, even though Catherine will try to stop him?
4. Two players take turns entering a symbol in an empty cell of a $1 \times n$ chessboard, where n is an integer greater than 1. Aaron always enters the symbol X and Betty always enters the symbol O. Two identical symbols may not occupy adjacent cells. A player without a move loses the game. If Aaron goes first, which player has a winning strategy?
5. Attached to each of a number of objects is a tag which states the correct mass of the object. The tags have fallen off and have been replaced on the objects at random. We wish to determine if by chance all tags are in fact correct. We may use exactly once a horizontal lever which is supported at its middle. The objects can be hung from the lever at any point on either side of the support. The lever either stays horizontal or tilts to one side. Is this task always possible?
6. The audience arranges n coins in a row. The sequence of heads and tails is chosen arbitrarily. The audience also chooses a number between 1 and n inclusive. Then the assistant turns one of the coins over, and the magician is brought in to examine the resulting sequence. By an agreement with the assistant beforehand, the magician tries to determine the number chosen by the audience.
   (a) Prove that if this is possible for some n, then it is also possible for $2n$.
   (b) Determine all n for which this is possible.
7. For each letter in the English alphabet, William assigns an English word which contains that letter. His first document consists only of the word assigned to the letter A. In each subsequent document, he replaces each letter of the preceding document by its assigned word. The fortieth document begins with "Till whatsoever star that guides my moving." Prove that this sentence reappears later in this document.

Note: The problems are worth 5, 3+3, 6, 7, 8, 4+5 and 9 points respectively.