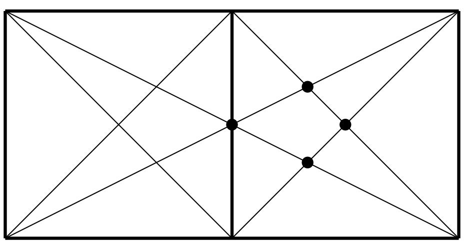
International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2006.**

1. A pool table has a shape of a 2×1 rectangle; there are six pockets: one in each corner, and one in the midpoint of each of the long sides of the table. What is the minimal number of balls one needs to put on the table so that every pocket lies on the same line with at least two balls? (Consider pockets and balls as points.) (B.R. Frenkin)

   ANSWER. 4 balls.

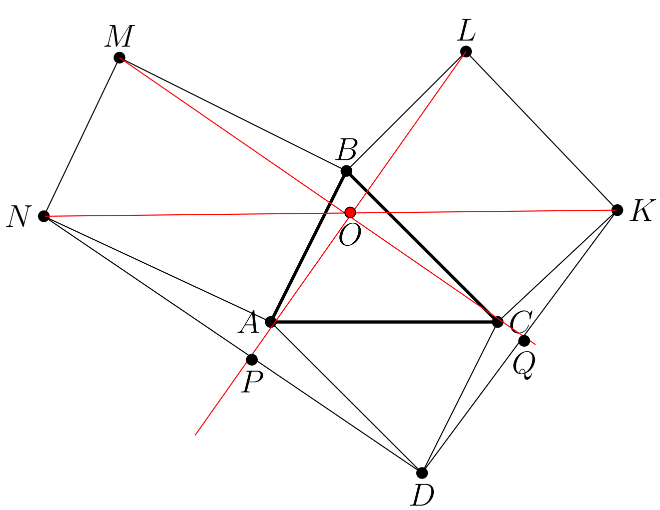
   

   SOLUTION. The example for 4 balls is given on the picture. Let us show that 3 balls are not enough. Straight line passing through two balls inside the rectangle intersects its boundary at exactly two points. We have 6 pockets, so we need at least 3 straight lines. Three balls creates three lines if and only if these balls form a triangle. However, all possible straight lines are drawn on the picture and none of them form a triangle with vertices inside the pool table.

2. Prove that one can find 100 pairs of integers with the following property: in the decimal representation of each integer, each digit is greater or equal to 6, and the product of the two integers in the pair is also an integer whose decimal representation has no digits less than 6. (S.I. Tokarev, A.V.Shapovalov)

   SOLUTION. All pairs (7, 9...97) are in use to our problem since their products are equal to 67...79.

3. Assume an acute triangle ABC is given. Two equal rectangles, $ABMN$ and $LBCK$, are drawn on the sides AB and BC in the outside. Given that $AB = LB$, prove that the straight lines AL, NK, and MC are concurrent. (A.Gavriluk)

   SOLUTION 2. Draw a parallelogram $ABCD$. Then $ALKD$ and $CDNM$ are also parallelograms. Isosceles $\triangle CBM$ can be obtained from $\triangle ABL$ by the rotation by 90° and homothety, thus $CM \perp AL$, but then and $CM \perp KD$. Continuation of MC, height CQ in the isosceles triangle KCD is its median, consequently CM is the perpendicular from the midpoint of KD. Similarly AL is the perpendicular from the midpoint of ND. Parallelogram $OPDQ$ is a rectangle, hence triangle KDN is right-angled, and perpendiculars from the midpoints of its legs pass through the midpoint of the hypotenuse KN.

Solution 1. Consider escribed circles for the given rectangles. Denote their second point of intersection by O. Then $\angle BON = \angle BOK = 90°$. Hence points N, O, K are situated on the straight line perpendicular to BO. Observe that angles NBA and LBK are equal (since corresponding triangles are equal). Since angles leaning on one edge are equal, we get the equalities: $\angle NOA = \angle NBA = \angle LBK = \angle LOK$, consequently points A, O, L are also situated on a straight line. Similarly points M, C, O are situated on a straight line. Thus O is the common point of these straight lines. Remark. Perpendicularity of AL and CM can be proved without rotating homothety, just using angles counting. Assume $\angle ABC = b$. In the isosceles triangles ABL and MBC angles B are equal to $b + 90°$, consequently other angles are equal to $45° - b/2$. This means that

$$\angle AOC = 180° \angle OAC \angle OCA = 180° \angle BAC \angle BCA + 2(45° - b/2) = \angle ABC + 2(45° - b/2) = 90°.$$

4. Does there exists a positive integer n such that the leftmost digit in the decimal representation of 2^n is 5, while the leftmost digit in the decimal representation of 5^n is 2? (G.A.Galperin)

   Solution. No. Observe that $2^n \cdot 5^n = 10^n$. If in decimal notations of 2^n and 5^n we change all digits except first for zeros each number decreases but no more than in two times. Product of new numbers will be less than 10^n, but not greater than in 4 times, therefore, it is not equal to $10\ldots0$. However if one of the changed numbers had leftmost digit 5, while other one had leftmost digit 2, then product would be equal to $50\ldots0 \cdot 20\ldots0 = 10\ldots0$. Contradiction.

5. Rectangular table of the size 2005×2006 is filled with integers 0, 1, and 2 in such a way that the sum of integers in each row and each column of the table is divisible by 3. What is the maximal number of 1's in such a configuration? (I.I. Bogdanov)

   Solution. Assume there are n zeros and d twos in the table. We have 2005 rows of the length 2006 and 2006 columns of the length 2005. In order to the sum of integers in a row be divisible by 3 there should be at least one two or at least two zeros. Hence $d + n/2 \geq 2005$. Similarly, there should be at least one zero or two twos in each column, consequently $n + d/2 \geq 2006$. Summing these inequalities and dividing by 3/2 we obtain $n + d \geq 2674$, i.e. the number if ones is not greater than $2005 \cdot 2006 - 2674$.

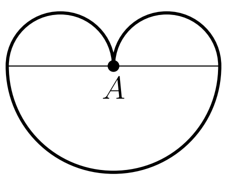
0	0	1	1	...	1	1
1	1	0	0	...	1	1
...	...	...	...	...	...	...
1	1	1	1	...	2	1
1	1	1	1	...	2	1
1	1	1	1	...	1	2
1	1	1	1	...	2	1

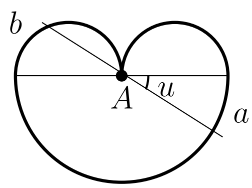
Now consider the table with $n = 1338$ and $d = 1336$. Arrange 1338 zeros in horizontal pairs beginning from the upper left corner (in 669 rows and 1338 columns) and 1336 twos in vertical pairs beginning from the lower right corner (in 1336 rows and 668 columns) and fill all the rest squares with ones (look at the picture to the right). Since 669+1336=2005 and 1338+668=2006 there will be zeros and twos in each row and each column and their amounts will be right for making sums of each row and column divisible by 3. So the answer is: the maximum number of ones is equal to $2005 \cdot 2006 - 2674 = 4022030$.

6. A curvilinear polygon is by definition a polygon whose edges are circle arcs. Does there exist a curvilinear polygon P and a point A on its boundary such that every line passing through A divides the boundary of the polygon into two pieces of equal total length? (S.V. Markelov)

SOLUTION. Yes, it exists, consider the upper picture. Take an arbitrary segment which has A as its midpoint and draw half of the circle with this segment as diameter, and at the other side of the segment two halves of the circles with halves of the segment as diameters. The perimeter of the figure is equal to the twice the length of the smaller circle.

It is obvious that initial segment divides the perimeter into two equal parts. Draw any other straight line through the point A denote the angle between this straight line and initial segment by u (measured in radians; look at the lower picture). The length of the upper part upper decreased by the length of the arc a and increased by the length of the arc b. We are going to prove that these lengths are equal. Denote by r the radius of the smaller circle. Since we have the inscribed angle $b = 2ur$. Larger circles radius is $2r$, but the angle for it is central, this means that $a = u \cdot 2r$.





7. George and Jake are each given an identical copy of a 5×5 table filled with 25 pairwise different integers. George chooses the maximal integer in his table, then deletes the row and the column which contain this integer, then chooses the maximal integer in the remaining 4×4 table, then deletes the row and the column, and so on. Jake does the same, but each time he chooses the minimal integer, not the maximal one. Can it be that in the end, the sum of the 5 integers chosen by Jake is

   (a) greater than the sum of the 5 integers chosen by George?

   (b) greater than the sum of any other 5 integers from the original table chosen so that no two of them lie in the same row, not in the same column?

(S.I.Tokarev, A.Y. Avnin)

SOLUTION. (a) No, it cant. Denote the Georges numbers in the order of their selection by b_0, b_1, b_2, b_3, b_4, and Jakes ones by m_0, m_1, m_2, m_3, m_4. Let us show that if $i + j < 5$, then $b_i \geq m_j$. Number is equal to the amount of rows and columns erased when the number is being chosen. For example, when we are choosing b_1 one row and one column is erased, while for m_3 3 rows and 3 columns. Summing, we obtain that the amount of erased rows and columns is not greater than 4, thus at least one number a was not erased in both cases. George was choosing the maximum numbers, consequently $b_1 \geq a$. Jake was choosing the minimum ones and $a \geq m_3$. Hence $b_1 \geq m_3$. Similarly $b_0 \geq m_4$, $b_2 \geq m_2$, $b_3 \geq m_1$, $b_4 \geq m_0$. This means that Georges sum is not less than Jakes one.

(b) SOLUTION 1. Yes, it can. Consider the left table:

10000	1001	1002	1003	1004
1005	**1000**	101	102	103
1006	104	**100**	11	12
1007	105	13	**10**	2
1008	106	14	3	**1**

111	210	310	410	510
120	221	320	420	520
130	230	331	430	530
140	240	340	441	540
150	250	350	450	551

Here the sum of the numbers chosen by Jake is equal to 11111 (they are marked). Let us show that it is not possible to obtain greater sum. If we do not take the number 10000, then the sum will be less than $1008 \cdot 5 = 5040$, so we have to take it 10000. Similarly after taking 10000 we have to take 1000, then 100, then 10 and finally 1. As a result we obtain Jakes numbers.

Solution. 2. Yes, it can. Consider the right table:

Let us add in the column numbers from any admissible collection. At the leftmost digit sum is equal to the number of diagonal numbers in the collection. At the tens digit for any collection its sum is equal to 1+2+3+4+5 (this digit depends on the row and we have members from each row). Similarly, at the hundreds digit the sum is equal to 1+2+3+4+5 (this digit depend only on the column). Thus the collection of diagonal numbers has the maximum sum. But it is evident that Jake chooses them.