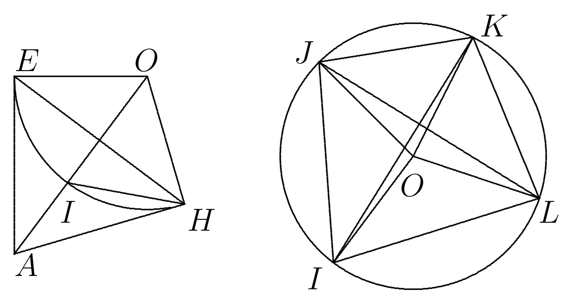
International Mathematics

TOURNAMENT OF THE TOWNS.
Solutions

Senior O-Level Paper **Fall 2006**[1]

1. We claim that the sum of the numbers in Mary's notebook is equal to the product of the three numbers originally on the blackboard. We use induction on the number n of steps for Mary to reduce one of the numbers to 0. For $n = 1$, one of the three numbers on the blackboard must be equal to 1 and is reduced to 0. In recording the product of the other two numbers, Mary is in fact recording the product of all three numbers. Suppose the claim holds for some $n \geq 1$. Let the original numbers be x, y and z. By symmetry, we may assume that Mary records xy in her notebook and replaces z by $z - 1$. By the induction hypothesis, the sum of the remaining numbers in her notebook is equal to $xy(z - 1)$, so that the sum of all the numbers in her notebook is equal to $xy + xy(z - 1) = xyz$.

2. Let O be the incentre of $ABCD$. Let AO intersect the incircle of $ABCD$ at I. Let $\angle AOH = \angle AOE = 2\alpha$. Since $\angle AHO = 90° = \angle AEO$, A, E, O and H are concyclic, so that $\angle AHE = \angle AOE = 2\alpha$. We have $\angle OAH = 180° - \angle AOH - \angle AHO = 90° - 2\alpha$ and since $OH = OI$, $\angle OIH = \frac{1}{2}(180° - \angle IOH) = 90° - \alpha$. It follows that $\angle AHI = \angle OIH - \angle OAH = \alpha = \frac{1}{2}\angle AHO$. Hence I is the incentre of triangle HAE. Similarly, the respective incentres J, K and L of triangles EBF, FCG and GDH all lie on the incircle of $ABCD$. Let $\angle BOE = \angle BOF = 2\beta$, $\angle COF = \angle COG = 2\gamma$ and $\angle DOG = \angle DOH = 2\delta$. Then $\angle IOJ + \angle KOL = \alpha + \beta + \gamma + \delta = 180°$. Now $\angle ILJ + \angle KIL = \frac{1}{2}(\angle IOJ + \angle KOL) = 90°$. Hence IK and JL are perpendicular to each other.



3. We can replace each number by the remainder obtained when it is divided by 4. Thus we have 1003^2 copies of each of 0, 1, 2 and 3. Divide the board into 1003^2 2×2 subboards. Each subboard may contain at most one 0 and at most one 2. Since we have exactly as many copies of each number as we have subboards, there is exactly one 0 and exactly one 2 in each subboard. The remaining two cells in each subboard must both contain copies of 1 or both contain copies of 3. However, this is impossible as we have an odd number of copies of each of 1 and 3.

[1]Courtesy of Professor Andy Liu.

4. Let the first term and the common difference of the arithmetic progression be a and $d > 0$ respectively. Let the first term and the common ratio of the geometric progression be b and $r > 1$ respectively. Then $b = a + id$, $br = a + jd$ and $br^2 = a + kd$ for some integers i, j and k such that $0 \leq i < j < k$. It follows that $b(r-1) = (j-i)d$ and $br(r-1) = (k-j)d$, so that $r = \frac{k-j}{j-i}$ is a rational number. Let $t = \frac{a}{d}$. From $a + jd = br = r(a + id)$, we have $t + j = rt + ri$. Hence $t = \frac{j-ri}{r-1}$ is also rational. Divide all the terms of both progressions by d. Then the arithmetic progression has first term t and common difference 1 while the geometric progression has first term $\frac{b}{d}$ and common ratio r. Let $t = \frac{p}{q}$ where p and q are relatively prime positive integers. Then all terms in the arithmetic progression are of the form $\frac{p+kq}{q}$ for some non-negative integer k. If r is not an integer, then when n is a sufficiently large positive integer, the expression of $\frac{b}{d}r^n$ as a fraction in the simplest terms will have a denominator greater than q. This contradicts the hypothesis that every term of the geometric progressoin is a term of the arithmetic progression.

5. The task is possible. Let the side length of the cube be 4. In the diagram below, each of U, V, W, X, Y and Z is at a distance 1 from the nearest vertex of the cube. Clearly, UWX and VYZ are equilateral triangles with side length $3\sqrt{2}$. Note that $\angle EAW = 90° = \angle WEZ$. Hence $WZ = \sqrt{EZ^2 + EA^2 + WA^2} = 3\sqrt{2}$ also. By symmetry, WV, XV, XY, UY and UZ all have the same length. It follows that $UVWXYZ$ is indeed a regular octahedron.