Seniors

(Grades 11 and 12)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2006.**[2]

1 [**4**] When Ann meets new people, she tries to find out who is acquainted with who. In order to memorize it she draws a circle in which each person is depicted by a chord; moreover, chords corresponding to acquainted persons intersect (possibly at the ends), while the chords corresponding to non-acquainted persons do not. Ann believes that such set of chords exists for any company. Is her judgement correct?

2 [**6**] Suppose ABC is an acute triangle. Points A_1, B_1 and C_1 are chosen on sides BC, AC and AB respectively so that the rays A_1A, B_1B and C_1C are bisectors of triangle $A_1B_1C_1$. Prove that AA_1, BB_1 and CC_1 are altitudes of triangle ABC.

3 [**6**] The n-th digit of number $a = 0.12457\ldots$ equals the first digit of the integer part of the number $n\sqrt{2}$. Prove that a is irrational number.

4 [**6**] Is it possible to split a prism into disjoint set of pyramids so that each pyramid has its base on one base of the prism, while its vertex on another base of the prism ?

5 [**7**] Let $1 + 1/2 + 1/3 + \cdots + 1/n = a_n/b_n$, where a_n and b_n are relatively prime. Show that there exist infinitely many positive integers n, such that $b_{n+1} < b_n$.

6 Let us say that a deck of 52 cards is arranged in a “regular” way if the ace of spades is on the very top of the deck and any two adjacent cards are either of the same value or of the same suit (top and bottom cards regarded adjacent as well). Prove that the number of ways to arrange a deck in regular way is

a) [**3**] divisible by 12!

b) [**5**] divisible by 13!

7 Positive numbers $x_1, \ldots, x_k$ satisfy the following inequalities:

$$x_1^2 + \cdots + x_k^2 < \frac{x_1 + \cdots + x_k}{2} \quad \text{and} \quad x_1 + \cdots + x_k < \frac{x_1^3 + \cdots + x_k^3}{2}.$$

a) [**3**] Show that $k > 50$;

b) [**3**] Give an example of such numbers for some value of k;

c) [**3**] Find minimum k, for which such an example exists.

[2]Your total score is based on the three problems for which you earn the most points. Points for each problem are shown in brackets [ ].