International Mathematics

TOURNAMENT OF THE TOWNS.
Solutions

Junior O-Level Paper **Fall 2006**[1]

1. We claim that the sum of the numbers in Mary's notebook is equal to the product of the two numbers originally on the blackboard. We use induction on the number n of steps for Mary to reduce one of the numbers to 0. For $n = 1$, the two numbers on the blackboard must be equal to each other. In recording the square of the smaller number, Mary is in fact recording the product of the two numbers. Suppose the claim holds for some $n \geq 1$. Let the original numbers be x and y with $x < y$. Then Mary records x^2 in her notebook and replaces y by $y-x$. By the induction hypothesis, the sum of the remaining numbers in her notebook is equal to $x(y-x)$, so that the sum of all the numbers in her notebook is equal to $x^2+x(y-x) = xy$.

2. (a) Ask each of the three people: "Are you a Normal?" Since the Knight and the Knave will give opposite answers, the three answers consist of a matching pair and an odd one out. If the odd answer is "Yes", the replier is the Knight, and if the odd answer is "No", the replier is the Knave. From this person, we can learn the identity of all three people.

   (b) The first Normal will act as though he is a Knight while the second Normal will act as though he is a Knave. Then we cannot tell the difference between the first Normal and the Knight, nor between the second Normal and the Knave.

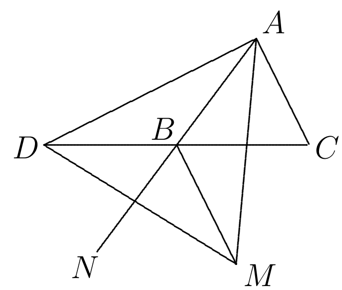
3. Suppose a number is expressible in the form $a^2 - b^2 = (a + b)(a - b)$. If a and b are of the same parity, then the product is divisible by 4. If they are of opposite parity, then the product is odd. Conversely, a number of the form $4n$ may be expressed as $(n + 1)^2 - (n - 1)^2$ while a number of the form $2n + 1$ may be expressed as $(n + 1)^2 - n^2$. Hence a number is not expressible in the form $a^2 - b^2$ if and only if it is of the form $4n + 2$. The only way in which a product takes the form $4n + 2$ is when exactly one of the factors is of that form, and the others are odd.

   (a) Suppose an even number of the 2007 numbers is of the form $4n+2$. Then there exists at least one number not of this form, and we choose this number. Suppose an odd number of the 2007 numbers is of the form $4n + 2$. Then we choose any of these. Among the remaining 2006 numbers, there will not be exactly one number of the form $4n+2$. Hence their product is expressible in the form $a^2 - b^2$.

   (b) If there is a number of the form $4n + 2$ other than 2006, then any of the other 2005 numbers may be chosen so that the product of the remaining 2006 numbers will not be of the form $4n + 2$. Hence the choice will not be unique. It follows that 2006 is the only number of the form $4n + 2$, and it must be the chosen number.

[1]Courtesy of Professor Andy Liu.

4. Note that $\angle NBD = \angle ABC$ and $\angle NBM = \angle CBM$. Hence $\angle DBM = \angle ABM$. Since we also have $BD = BA$ and $BM = BM$, triangles DBM and ABM are congruent, so that $\angle MDC = \angle MAN$. Now M is an excentre of triangle ABC. Hence $\angle MAN = \angle MAC$. From $\angle MDC = \angle MAC$, we can conclude that A, C, M and D are concyclic.



5. The maximum value of n is at most 8 because such a polygon can only have 8 possible orientations. We may use each of them once as otherwise we would have two copies which are parallel translates of each other. The maximum value is in fact 8 as it is attained by the polygon in the diagram below.