International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper[1] **Spring 2005.**

1. The graphs of four functions of the form $y = x^2 + ax + b$, where a and b are real coefficients, are plotted on the coordinate plane. These graphs have exactly four points of intersection, and at each one of them, exactly two graphs intersect. Prove that the sum of the largest and the smallest x-coordinates of the points of intersection is equal to the sum of the other two.

2. The base-ten expressions of all the positive integers are written on an infinite ribbon without spacing: 1234567891011.... Then the ribbon is cut up into strips seven digits long. Prove that any seven digit integer will:

   (a) appear on at least one of the strips;

   (b) appear on an infinite number of strips.

3. M and N are the midpoints of sides BC and AD, respectively, of a square $ABCD$. K is an arbitrary point on the extension of the diagonal AC beyond A. The segment KM intersects the side AB at some point L. Prove that $\angle KNA = \angle LNA$.

4. In a certain big city, all the streets go in one of two perpendicular directions. During a drive in the city, a car does not pass through any place twice, and returns to the parking place along a street from which it started. If it has made 100 left turns, how many right turns must it have made?

5. The sum of several positive numbers is equal to 10, and the sum of their squares is greater than 20. Prove that the sum of the cubes of these numbers is greater than 40.

Note: The problems are worth 3, 3+1, 4, 4 and 5 points respectively.

[1] Courtesy of Andy Liu.

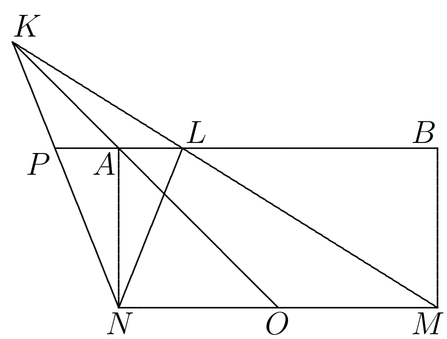
Solution to Senior O-Level Spring 2005

1. Let the parabolas be $y_i = x^2 + a_i x + b_i$, $1 \le i \le 4$. Now y_i and y_j intersect if and only if $a_i \ne a_j$, and if that it the case, they intersect at exactly one point with $x = \frac{b_i - b_j}{a_j - a_i}$. Since we have only four points of intersection, we must have two distinct values of a_i, each appearing twice. Hence we may assume that $a_2 = a_1$ and $a_4 = a_3$. By symmetry, we may assume that $b_1 < b_2$, $b_3 < b_4$ and $a_1 < a_3$. This means that y_1 is below y_2, y_3 is below y_4 and the common axis of y_1 and y_2 is to the right of the common axis of y_3 and y_4. It follows that the rightmost point of intersection is that of y_2 with y_3 while the leftmost point of intersection is that of y_1 with y_4. The sum of their x-coordinates is $\frac{b_1 - b_4}{a_3 - a_1} + \frac{b_2 - b_3}{a_3 - a_1} = \frac{b_1 + b_2 - b_3 - b_4}{a_3 - a_1}$. The sum of the x-coordinates of the other two points of intersections is $\frac{b_1 - b_3}{a_3 - a_1} + \frac{b_2 - b_4}{a_3 - a_1} = \frac{b_1 + b_2 - b_3 - b_4}{a_3 - a_1}$ as well.

2. (a) Suppose n is a seven-digit number. Consider the seven consecutive eight-digit numbers $10n$, $10n + 1$, $\dots$, , $10n + 6$. Since 7 and 8 are relatively prime, some strip will start with one of these numbers and n appears on it.

   (b) As in (a), we can consider the seven consecutive nine-digit numbers $100n$, $100n + 1$, $\dots$, $100n + 6$, the seven consecutive ten-digit numbers $1000n$, $1000n + 1$, $\dots$, $1000n + 6$, and so on. For each number of digits not divisible by 7, we get a strip on which n appears.

3. Let AC cut MN at O, and extend BA to cut KN at P. Since PL is parallel to NM and O is the midpoint of NM, A is the midpoint of AL. Hence triangles PAN and LAN are congruent to each other, so that $\angle KNA = \angle LNA$.



4. In tracing a simple closed curve, the net change in the direction of the car is 360°, clockwise or counterclockwise. Hence it must have made 96 or 104 right turns.

5. Suppose $a_1 + a_2 + \cdots + a_n = 10$ and $a_1^2 + a_2^2 + \cdots + a_n^2 > 20$. By Cauchy's Inequality,

$$\begin{aligned} 10(a_1^3 + a_2^3 + \cdots + a_n^3) &= (a_1 + a_2 + \cdots + a_n)(a_1^3 + a_2^3 + \cdots + a_n^3) \\ &\ge (\sqrt{a_1}\sqrt{a_1^3} + \sqrt{a_2}\sqrt{a_2^3} + \cdots + \sqrt{a_n}\sqrt{a_n^3})^2 \\ &= (a_1^2 + a_2^2 + \cdots + a_n^2)^2 \\ &> 400. \end{aligned}$$

   Hence $a_1^3 + a_2^3 + \cdots + a_n^3 > 40$.