**International Mathematics**
**TOURNAMENT OF THE TOWNS**

**Senior A-Level Paper**[1] **Spring 2005.**

1. On the graph of a polynomial with integral coefficients are two points with integral coordinates. Prove that if the distance between these two points is integral, then the segment connecting them is parallel to the $x$-axis.

2. A circle $\omega_1$ with centre $O_1$ passes through the centre $O_2$ of a second circle $\omega_2$. The tangent lines to $\omega_2$ from a point $C$ on $\omega_1$ intersect $\omega_1$ again at points $A$ and $B$ respectively. Prove that $AB$ is perpendicular to $O_1O_2$.

3. John and James wish to divide 25 coins, of denominations 1, 2, 3, ..., 25 kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. John makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kokeps wins. Which player has a winning strategy?

4. For any function $f(x)$, define $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for any integer $n \geq 2$. Does there exist a quadratic polynomial $f(x)$ such that the equation $f^n(x) = 0$ has exactly $2^n$ distinct real roots for every positive integer $n$?

5. Prove that if a regular icosahedron and a regular dodecahedron have a common circumsphere, then they have a common insphere.

6. A *lazy* rook can only move from a square to a vertical or a horizontal neighbour. It follows a path which visits each square of an $8 \times 8$ chessboard exactly once. Prove that the number of such paths starting at a corner square is greater than the number of such paths starting at a diagonal neighbour of a corner square.

7. Every two of 200 points in space are connected by a segment, no two intersecting each other. Each segment is painted in one colour, and the total number of colours is $k$. Peter wants to paint each of the 200 points in one of the colours used to paint the segments, so that no segment connects two points both in the same colour as the segment itself. Can Peter always do this if

   (a) $k = 7$;

   (b) $k = 10$?

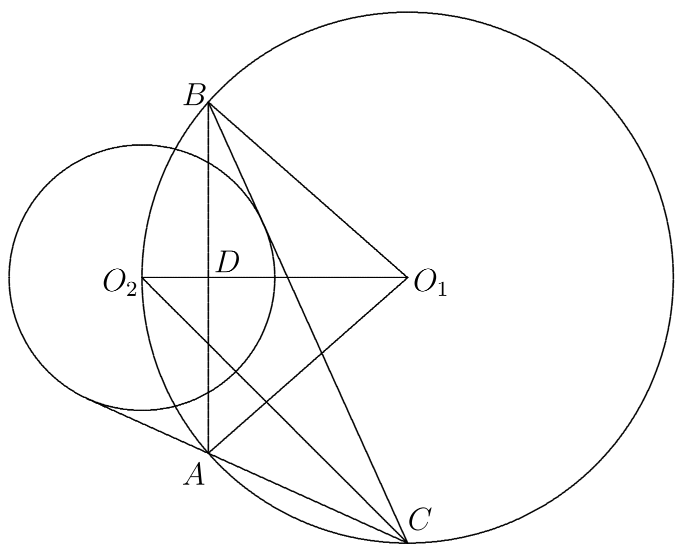
**Note:** The problems are worth 4, 5, 5, 6, 7, 7 and 4+4 points respectively.

[1] Courtesy of Andy Liu.

**Solution to Senior A-Level Spring 2005**

1. Let $f(x)$ be a polynomial with integral coefficients such that $f(x_1)$ and $f(x_2)$ are integers for some integers $x_1$ and $x_2$. Since $x_1^k - x_2^k$ is divisible by $x_1 - x_2$ for all $k$, $f(x_1) - f(x_2) = n(x_1 - x_2)$ for some integer $n$. If in addition the distance between the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is also an integer $m$, then $(x_1 - x_2)^2 + (f(x_1) - f(x_2))^2 = m^2$. Then $(x_1 - x_2)^2(1 + n^2) = m^2$, so that $1 + n^2$ is also the square of an integer. This is only possible for $n = 0$. Hence $f(x_1) - f(x_2) = 0$, so that $f(x_1) = f(x_2)$, and the line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$ is indeed parallel to the $x$-axis.

2. Since $CA$ and $CB$ are tangents to $\omega_2$, we have $\angle ACO_2 = \angle BCO_2$. It follows that we have $\angle AO_1O_2 = 2\angle ACO_2 = 2\angle BCO_2 = \angle BO_1O_2$. Moreover, $O_1A = O_1B$ and $O_1D = O_1D$, where $D$ is the point of intersection of $AB$ and $O_1O_2$. It follows that triangles $O_1AD$ and $O_1BD$ are congruent. Hence $\angle ADO_1 = \angle BDO_1$. Since their sum is 180°, each is 90° and $O_1O_2$ is indeed perpendicular to $AB$.

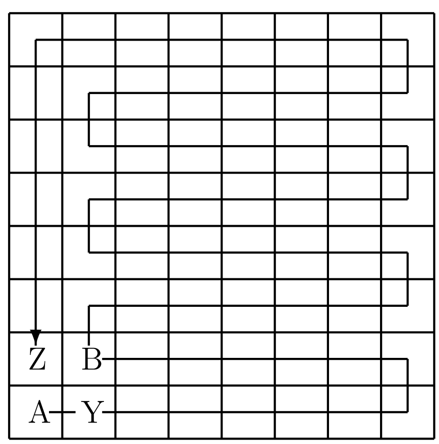



3. James can always get more kopeks than John. Upon John's initial offer, James can either take it or leave it. If there is a way for him to get more kopeks than John by taking it, there is nothing further to prove. If there are no ways, then he makes John take it, and there are no ways for John to get more kopeks than he.

4. Such a function is $f(x) = x^2 - 2$. For $f(x) = 0$, we have $x^2 = 2$, and the roots are $\pm\sqrt{2}$. We claim that every root of $f^{n+1}(x) = 0$ has the form $r_{n+1} = \pm\sqrt{2 \pm r_n}$ for some root $\pm r_n$ of $f^n(x) = 0$. Indeed, $f^{n+1}(r_{n+1}) = f^n((\pm\sqrt{2 \pm r_n})^2 - 2) = f^n(\pm r_n) = 0$. Since the degree of $f^{n+1}(x)$ is double that of $f^n(x)$, these are all the roots. We prove by induction on $n$ that $\pm r_n$ are real and $|r_n| < 2$ for all $n$. For $n = 1$, this is certainly the case with $\pm\sqrt{2}$. Suppose the result holds for some $n \geq 1$. Since $|r_n| < 2$, $2 \pm r_n > 0$ so that $r_{n+1} = \pm\sqrt{2 \pm r_n}$ are real. Moreover, $|2 \pm r_n| \leq 2 + |r_n| < 4$, so that $|r_{n+1}| < 2$. Finally, observe that $\sqrt{2}$ and $-\sqrt{2}$ are distinct, and that distinct roots of $f^n(x) = 0$ lead to distinct roots of $f^{n+1}(x) = 0$.

5. Let $O$ be the circumcentre of the icosahedron, $C$ the centre of one of its faces and $A$ a vertex of that face. Its circumradius is $OA$, and its inradius is $OC$. Construct a dual dodecahedron by joining the centrers of adjacent faces of the icosahedron. Now $C$ is a vertex of three faces of this dodecahedron, and the centre $B$ of one of these faces lies on $OA$. Its circumradius is $OC$ and its inradius is $OB$. Note that in triangles $OAC$ and $OCB$, $\angle AOC = \angle COB$ and $\angle OCA = 90° = \angle OBA$. Hence they are similar to each other, so that $\frac{OA}{OC} = \frac{OC}{OB}$. If we rescale the two solids so that their circumradii are equal, then so are their inradii.

6. The diagram below shows a path from A to Z along which a lazy rook visits every square of the $8 \times 8$ chessboard once and only once, where A, B, Y and Z are as labelled. Note that A and B have the same colour in the usual chessboard pattern. Since the squares visited by the lazy rook must alternate in colour, no path can start from A and end at B, or vice versa. We claim that there are more such paths starting from A than those starting from B. For each path starting from B, since the path cannot end at A, the lazy rook must visit A between visits to Y and Z. Suppose the lazy rook visits Y first. Then the path corresponds to the following one starting from A: move to Y, follow the original path in reverse to B, move to Z, and follow the original path to the end. If the lazy rook visits Z first, then start from A, move to Z, follow the original path in reverse to B, move to Y, and follow the original path to the end. The path in the diagram below does not correspond to any path starting from B because no path starting from B can end at Z unless it moves from A to Z. This justifies our claim.



7. (a) Peter cannot always do so when $k = 7$, even when there are only 128 points. We ignore the remaining 72 points and segments joining them to one another or to our 128 points. Divide the 128 points into 64 pairs, and paint the segments joining the two points in each pair red. Combine the 64 pairs into 32 quartets. In each quartet, all segments joining one point from each pair are painted blue. Combine the 32 quartets into 16 octets. In each octet, all segments joining one point from each quartet are painted yellow. Combine the 16 octets into 8 hexidecatets. In each hexidecatets, all segments joining one point from each octet are painted green. Combine the 8 hexidecatets into 4 groups. In each group, all segments joining one point from each hexidecatet are painted orange. Combine the 4 groups into 2 halves. In each half, all segments joining one point from each group are painted violet. Finally, combine the 2 halves into 1 set. In the set, all segments joining one point from each half are painted black. Now Peter cannot have a black point in each half. Hence there is a half with no black points. Discard the other half. Now Peter cannot have a violet point in each group. Hence there is a group with no violet points. Discard the other group. Now Peter cannot have an orange point in each hexidecatet. Hence there is a hexidecatet with no orange points. Discard the other hexidecatet. Now Peter cannot have a green point in each octet. Hence there is an octet with no green points. Discard the other octet. Now Peter cannot have a yellow point in each quartet. Hence there is a quartet with no yellow points. Discard the other quartet. Now Peter cannot have a blue point in each pair. Hence there is a pair with no belue points. Discard the other pair. In the remaining pair, both points are red and they are joined by a red segment.

   (b) **Solution by Cheng-Chiang Tasi, Kaohsiung High School, Taiwan.**
   Peter still cannot do so when $k = 10$, even when there are only 121 points. We ignore the remaining 79 points and segments joining them to one another or to our 121 points. We construct a finite geometry based on arithmetic modulo 11. Each point is given coordinates $(i, j)$, where each of $i$ and $j$ is an integer between 0 and 10 inclusive. Consider two points $(i_1, j_1)$ and $(i_2, j_2)$. If $i_1 = i_2$, the segment joining them is vertical. If $j_i = j_2$, the segment joining them is horizontal. We either paint such segments arbitrarily or leave them unpainted. In all other cases, the segment joining the two points has slope $m$, where $m$ is an integer between 1 and 10 inclusive. We paint such a segment in the $m$-th colour. Now Peter paint the 121 points in 10 colours. By the Pigeonhole Principle, there must be at least 13 points of the same colour, say the $m$-th one. Now there are 11 lines in this geometry with slope $m$, each passing through exactly 11 points. By the Pigeonhole Principle again, at least 2 of these 13 points must be on the same line. Then we have 2 points in the $m$-th colour, joined by a segment also in the $m$-th colour.