International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper[1] **Spring 2005.**

1. On the graph of a polynomial with integral coefficients are two points with integral coordinates. Prove that if the distance between these two points is integral, then the segment connecting them is parallel to the x-axis.

2. A circle ω_1 with centre O_1 passes through the centre O_2 of a second circle ω_2. The tangent lines to ω_2 from a point C on ω_1 intersect ω_1 again at points A and B respectively. Prove that AB is perpendicular to O_1O_2.

3. John and James wish to divide 25 coins, of denominations 1, 2, 3, ..., 25 kopeks. In each move, one of them chooses a coin, and the other player decides who must take this coin. John makes the initial choice of a coin, and in subsequent moves, the choice is made by the player having more kopeks at the time. In the event that there is a tie, the choice is made by the same player in the preceding move. After all the coins have been taken, the player with more kokeps wins. Which player has a winning strategy?

4. For any function $f(x)$, define $f^1(x) = f(x)$ and $f^n(x) = f(f^{n-1}(x))$ for any integer $n \geq 2$. Does there exist a quadratic polynomial $f(x)$ such that the equation $f^n(x) = 0$ has exactly 2^n distinct real roots for every positive integer n?

5. Prove that if a regular icosahedron and a regular dodecahedron have a common circumsphere, then they have a common insphere.

6. A *lazy* rook can only move from a square to a vertical or a horizontal neighbour. It follows a path which visits each square of an 8×8 chessboard exactly once. Prove that the number of such paths starting at a corner square is greater than the number of such paths starting at a diagonal neighbour of a corner square.

7. Every two of 200 points in space are connected by a segment, no two intersecting each other. Each segment is painted in one colour, and the total number of colours is k. Peter wants to paint each of the 200 points in one of the colours used to paint the segments, so that no segment connects two points both in the same colour as the segment itself. Can Peter always do this if

   (a) $k = 7$;

   (b) $k = 10$?

Note: The problems are worth 4, 5, 5, 6, 7, 7 and 4+4 points respectively.

[1]Courtesy of Andy Liu.