International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper[1] **Spring 2005.**

1. Anna and Boris move simultaneously towards each other, from points A and B respectively. Their speeds are constant, but not necessarily equal. Had Anna started 30 minutes earlier, they would have met 2 kilometers nearer to B. Had Boris started 30 minutes earlier instead, they would have met some distance nearer to A. Can this distance be uniquely determined?

2. Prove that one of the digits 1, 2 and 9 must appear in the base-ten expression of n or $3n$ for any positive integer n.

3. There are eight identical Black Queens in the first row of a chessboard and eight identical White Queens in the last row. The Queens move one at a time, horizontally, vertically or diagonally by any number of squares as long as no other Queens are in the way. Black and White Queens move alternately. What is the minimal number of moves required for interchanging the Black and White Queens?

4. M and N are the midpoints of sides BC and AD, respectively, of a square $ABCD$. K is an arbitrary point on the extension of the diagonal AC beyond A. The segment KM intersects the side AB at some point L. Prove that $\angle KNA = \angle LNA$.

5. In a certain big city, all the streets go in one of two perpendicular directions. During a drive in the city, a car does not pass through any place twice, and returns to the parking place along a street from which it started. If it has made 100 left turns, how many right turns must it have made?

Note: The problems are worth 3, 4, 5, 5 and 5 points respectively.

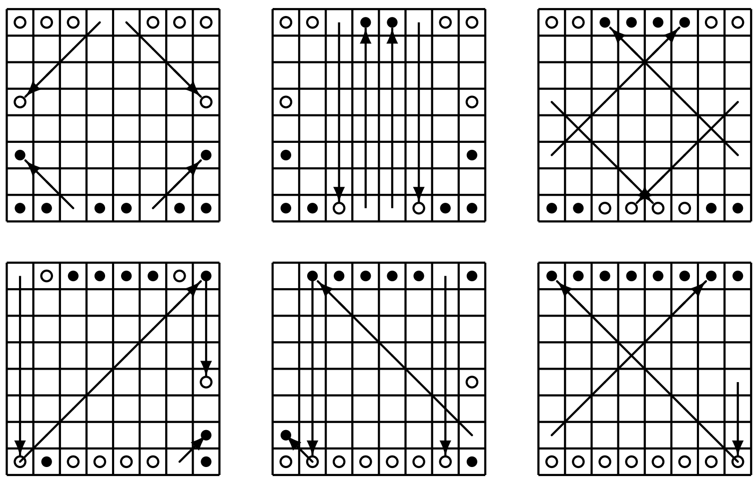
[1]Courtesy of Andy Liu.

Solution to Junior O-Level Spring 2005

1. Let the distance AB be x kilometres. Let the speeds of Anna and Boris be a and b kilometres respectively. Then the distance covered by Anna is $\frac{ax}{a+b}$ and that by Boris $\frac{bx}{a+b}$. When Anna covers 2 more kilometres and Boris 2 less, the difference in time spent is $\frac{1}{2}$ hours. It follows that $\frac{1}{a}(\frac{ax}{a+b}+2)-\frac{1}{b}(\frac{bx}{a+b}-2)=\frac{1}{2}$, which simplifies to $\frac{1}{a}+\frac{1}{b}=\frac{1}{4}$. Since this expression is symmetric, the two of them will meet 2 kilometres closer to A when Boris starts 30 minutes early.

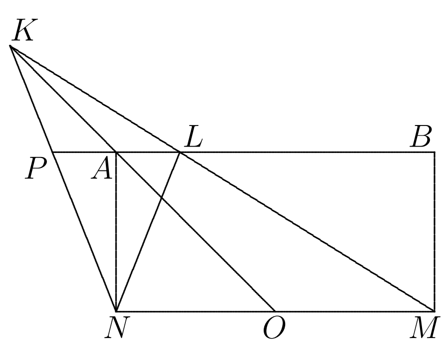
2. If the leading digit of n is 1, 2 or 9, there is nothing to prove. If it is 3, then the leading digit of $3n$ is either 9 or 1. If the leading digit of n is 4 or 5, the leading digit of $3n$ will be 1. If it is 6, then the leading digit of $3n$ is either 1 or 2. If the leading digit of n is 7 or 8, the leading digit of $3n$ will be 2. All cases have been covered, and the desired conclusion follows.

3. We first show that the task can be accomplished in 23 moves.



We now prove that we need at least 23 moves. Each the 16 Queens must move at least once. Of the two Queens on each inside column, at most one can move only once. This means at least 6 extra moves. Of the four Queens at the corners, at most three can move only once. This means at least 1 extra move. Hence the minimum is 23 moves.

4. Let AC cut MN at O, and extend BA to cut KN at P. Since PL is parallel to NM and O is the midpoint of NM, A is the midpoint of AL. Hence triangles PAN and LAN are congruent to each other, so that $\angle KNA = \angle LNA$.



5. In tracing a simple closed curve, the net change in the direction of the car is 360°, clockwise or counterclockwise. Hence it must have made 96 or 104 right turns.