

**International Mathematics
TOURNAMENT OF THE TOWNS**

Senior A-Level Paper

Fall 2005.

1. For which positive integers n can one find distinct positive integers a_1, a_2, \dots, a_n such that $\frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_n}{a_1}$ is also an integer?
2. Each side of a polygon is longer than 100 centimetres. Initially, two ants are on the same edge of the polygon, at a distance 10 centimetres from each other. They crawl along the perimeter of the polygon, maintaining the distance of 10 centimetres measured along a straight line.
 - (a) Suppose the polygon is convex. Is it always possible for each point on the perimeter of the polygon to be visited by both ants?
 - (b) Suppose the polygon is not necessarily convex. Is it always possible for each point on the perimeter of the polygon to be visited by at least one of the ants?
3. Initially, there is a rook on each of the 64 squares of an 8×8 chessboard. Two rooks attack each other if they are in the same row or column, and there are no other rooks directly in between. In each move, one may take away any rook which attacks an odd number of other rooks still on the chessboard. What is the maximum number of rooks that can be removed?
4. On a circle are a finite number of red points. Each is labelled with a positive number less than or equal to 1. The circle is to be divided into three arcs so that each red point is in exactly one of them. The sum of the labels of all red points in each arc is computed. This is taken to be 0 if the arc contains no red points. Prove that it is always possible to find a division for which the sums on any two arcs will differ by at most 1.
5. In triangle ABC , $\angle A = 2\angle B = 4\angle C$. Their bisectors meet the opposite sides at D , E and F respectively. Prove that $DE = DF$.
6. A blackboard is initially empty. In each move, one may either add two 1s, or erase two copies of a number n and replace them with $n - 1$ and $n + 1$. What is the minimum number of moves needed to put 2005 on the blackboard?

Note: The problems are worth 3, 2+3, 5, 6, 7 and 8 points respectively.

Solution to Senior A-Level Fall 2005

1. For $n = 1$, we may take $a_1 = 1$ and $\frac{a_1}{a_1} = 1$ is indeed an integer. For $n = 2$, consider any $a_1 < a_2$. We may assume that they are relatively prime to each other. Then $\frac{a_1}{a_2} + \frac{a_2}{a_1} = \frac{a_1^2 + a_2^2}{a_1 a_2}$. This is never an integer since the factor a_2 in the denominator cannot be cancelled out. For $n \geq 3$, take $a_k = (n - 1)^{k-1}$ for $1 \leq k \leq n$. These are distinct integers since $n - 1 > 1$. The desired sum is $\frac{1}{n-1} + \frac{1}{n-1} + \dots + \frac{1}{n-1} + (n - 1)^{n-1} = (n - 1)^{n-1} + 1$, which is an integer. Hence $n = 2$ is the only impossible case.

2. (a) Let $ABCD$ be a rhombus with BD horizontal and less than 10 centimetres long. Then the segment XY joining the two ants is almost vertical. Let X be the ant initially closer to A and Y be the ant initially closer to C . Then X can never visit C while Y cannot visit A .

- (b) Modify the rhombus $ABCD$ by moving C vertically towards A until AC is less than 10 centimetres long. Then the segment XY joining the two ants is still almost vertical. If XY is initially on AB or AD , neither X nor Y can visit C . If XY is initially on CB or CD , neither X nor Y can visit A .

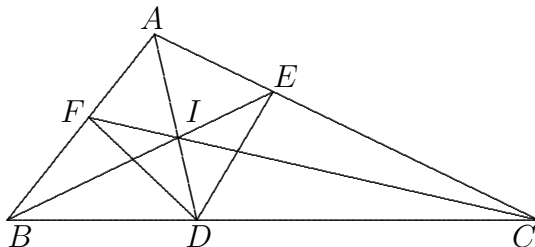
3. First, note that none of the corner rooks may be removed since each always attacks two other rooks. Moreover, we cannot leave behind only the four corner rooks, as otherwise the last to be taken away will attack two or zero other rooks. We can take away as many as $64 - 4 - 1 = 59$ rooks in two stages, as shown in the diagrams below.

•	1	2	3	4	5	6	•
13	○	○	○	○	○	○	7
14	15	16	17	18	19	○	8
20	21	22	23	24	25	○	9
26	27	28	29	30	31	○	10
32	33	34	35	36	37	○	11
38	39	40	41	42	43	•	12
•	44	45	46	47	48	49	•

•							•
	50	51	52	53	54	55	
						56	
						57	
						58	
						59	
						•	
•							•

4. For any arc A , denote by $f(A)$ the sum of the labels of the red points on A . Since there are finitely many red points, there are finitely many ways to divide them among three arcs. For each division, let the arcs be L , M and S , with $f(L) \geq f(M) \geq f(S)$. Choose among the divisions the one in which $f(L) - f(S)$ is minimum. We claim that for this division, $f(L) - f(S) \leq 1$. Suppose that this is not so. Now L and S are adjacent to each other. Let R be the red point on L closest to S and let r be its label. Consider the division (L', M', S') where the only change is that R moves from L to S . If $f(L) - r > f(S) + r$, then $f(L') = \max\{f(M), f(L) - r\}$ while $f(S') = \min\{f(M), f(S) + r\}$. On the other hand, if $f(L) - r \leq f(S) + r$, then $f(L') = \max\{f(M), f(S) + r\}$ while $f(S') = \min\{f(M), f(L) - r\}$. We have $f(L') - f(S') < f(L) - f(S)$ since $r \leq 1$, and $f(M)$ cannot be equal to $f(L)$ and $f(S)$ simultaneously. However, this contradicts our minimality assumption.

5. Let I be the incentre of triangle ABC . Let $\angle BCI = \angle ACI = \theta$. Then $\angle ABI = \angle CBI = 2\theta$ and $\angle CAI = \angle BAI = 4\theta$. Hence $\angle AIE = \angle BID = \angle BDI = 6\theta$, $\angle AIF = \angle AFI = 5\theta$ and $\angle AEI = 4\theta$. Let $AI = x$ and $DI = y$. Then $AF = IE = x$ and $BD = BI = x + y$. In triangle BAD , $\frac{AB}{AI} = \frac{DB}{DI}$, so that $BF = (\frac{DB}{DI} - \frac{AF}{AI})AI = \frac{x^2}{y}$. In triangle ABE , $\frac{EA}{EI} = \frac{BA}{BI}$, so that $AE = (\frac{AF+FB}{BI})EI = \frac{x^2}{y} = BF$. It follows that triangles EAD and FBD are congruent to each other, so that $DE = DF$.



6. **First Solution:**

Whenever we write n on the blackboard, we also write n^2 on a whiteboard, and whenever we erase n from the blackboard, we also erase n^2 from the whiteboard. Observe that when n appears in the smallest number of steps, it clearly comes from trading in two copies of $n - 1$. Hence $n - 1$ will not appear but $n - 2$ will. The step before involves trading in two copies of $n - 2$ for one of the $(n - 1)$ s, and so on. Hence the numbers on the blackboard other than n are $n - 2, n - 3, \dots, 3, 2, 1$, with an extra 1 if it is needed to make the total of the numbers even. Let $f(n)$ denote the minimum number of steps in order for the positive integer n to appear on the blackboard. In a move where we take two 1s, the sum of all the numbers on the whiteboard increases by $1+1=2$. In a move where we trade in two n s for $n + 1$ and $n - 1$, this sum also increases by $(n + 1)^2 + (n - 1)^2 - n^2 - n^2 = 2$. It follows that $f(n) = \frac{1}{2}(n^2 + (n - 2)^2 + (n - 3)^2 + \dots + 2^2 + 1^2 + 1^2)$ for $n \equiv 0, 1 \pmod{4}$, and $f(n) = \frac{1}{2}(n^2 + (n - 2)^2 + (n - 3)^2 + \dots + 2^2 + 1^2)$ for $n \equiv 2, 3 \pmod{4}$. In particular, $f(2005) = \frac{1}{2}(2005^2 + \frac{2003 \cdot 2004 \cdot 4007}{6} + 1) = 1342355520$.

Second Solution:

Let $f(n)$ denote the minimum number of steps in order for the positive integer n to appear on the blackboard. We claim that

$$f(n) - f(n - 1) = \begin{cases} \frac{n^2 - 2n + 4}{2} & \text{if } n \equiv 0 \pmod{4}; \\ \frac{n^2 - 2n + 3}{2} & \text{if } n \equiv 1 \pmod{2}; \\ \frac{n^2 - 2n + 2}{2} & \text{if } n \equiv 2 \pmod{4}. \end{cases}$$

From this, we have $f(n) - f(n - 4) = 2n^2 - 10n + 19$. Iterating this recurrence, we have

$$f(n) = \begin{cases} \frac{2n^3 - 3n^2 + 13n}{12} & \text{if } n \equiv 0, 1 \pmod{4}; \\ \frac{2n^3 - 3n^2 + 13n - 6}{12} & \text{if } n \equiv 2, 3 \pmod{4}. \end{cases}$$

In particular, $f(2005) = \frac{2005}{12}(2 \cdot 2005^2 - 3 \cdot 2005 + 13) = 1342355520$.

Observe that when n appears in the smallest number of steps, it clearly comes from trading in two copies of $n - 1$. Hence $n - 1$ will not appear but $n - 2$ will. The step before involves trading in two copies of $n - 2$ for one of the $(n - 1)$ s, and so on. Hence the numbers on the blackboard other than n are $n - 2, n - 3, \dots, 3, 2, 1$, with an extra 1 if it is needed to make the total of the numbers even. We now justify our claim. Consider first the case $n \equiv 0 \pmod{4}$. After $f(n - 1)$ steps, we have the numbers $1, 2, 3, \dots, n - 3, n - 1$ on the blackboard. To make $n - 2$ reappear, we take two 1s and trade upwards to obtain in succession two 2s, two 3s, and so on. After $n - 3$ steps, we have two $(n - 3)$ s, and we have $n - 2$ in $n - 2$ steps. Note that we have two 1s already. Thus it takes another $(n - 3) - 1$ steps to make $n - 3$ reappear. Since the number of 1s alternates between one and two, it takes $n - 4$ steps to make $n - 4$ reappear, $(n - 5) - 1$ steps to make $n - 5$ reappear, and so on. It follows that it takes altogether

$$(n - 2) + (n - 3) + \dots + 3 + 2 + 1 - \frac{n - 2}{2}$$

steps to obtain the numbers $1, 2, 3, \dots, n - 2, n - 1$. To make n appear, we take two 1s and trade upwards so that after n steps, we have the numbers $1, 1, 2, 3, \dots, n - 3, n - 2, n$. It follows that $f(n) - f(n - 1) = \frac{(n - 2)(n - 1)}{2} - \frac{n - 2}{2} + n = \frac{n^2 - 2n + 4}{2}$ as desired. For $n \equiv 1 \pmod{4}$, we have $f(n) - f(n - 1) = \frac{(n - 2)(n - 1)}{2} - \frac{n - 1}{2} + n = \frac{n^2 - 2n + 3}{2}$. For $n \equiv 2 \pmod{4}$, we have $f(n) - f(n - 1) = \frac{(n - 2)(n - 1)}{2} - \frac{n - 2}{2} + (n - 1) = \frac{n^2 - 2n + 2}{2}$. For $n \equiv 3 \pmod{4}$, we have $f(n) - f(n - 1) = \frac{(n - 2)(n - 1)}{2} - \frac{n - 3}{2} + (n - 1) = \frac{n^2 - 2n + 3}{2}$. Thus our claim is justified.