International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2005.**[1]

1. In triangle ABC, points D, E and F are the midpoints of BC, CA and AB respectively, while points L, M and N are the feet of the altitudes from A, B and C respectively. Prove that one can construct a triangle with the segments DN, EL and FM.

2. Each corner of a cube is labelled with a number. In each step, each number is replaced with the average of the labels of the three adjacent corners. All eight numbers are replaced simultaneously. After ten steps, all labels are the same as their respective initial values. Does it necessarily follow that all eight numbers are equal initially?

3. A segment of length 1 is cut into eleven shorter segments, each with length at most a. For what values of a will it be true that any three of the eleven segments can form a triangle, regardless of how the initial segment is cut?

4. A chess piece may start anywhere on a 15×15 chessboard. It can jump over 8 or 9 vacant squares either vertically or horizontally, but may not visit the same square twice. At most how many squares can it visit?

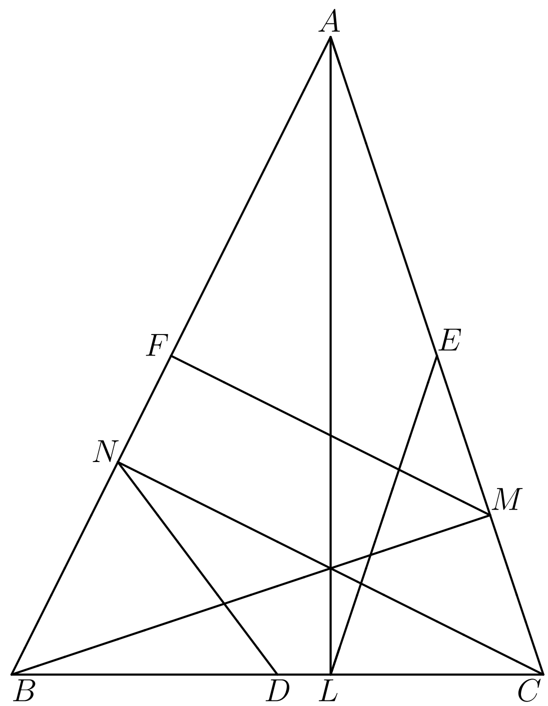
5. One of 6 coins is a fake. We do not know the weight of either a real coin or the fake coin, except that the real coins all weigh the same but different from the fake coin. Using a scale which shows the total weight of the coins being weighed, how can the fake coin be found in 3 weighings?

Note: The problems are worth 3, 3, 4, 4 and 5 points respectively.

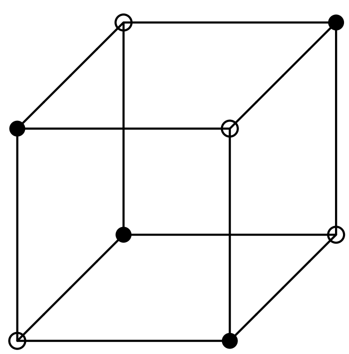
[1]Courtesy of Professor Andy Liu.

Solution to Junior O-Level Fall 2005

1. D is the midpoint of the hypotenuse of the right triangle NBC. Hence $DN = \frac{1}{2}BC$. Similarly, $EL = \frac{1}{2}CA$ and $FM = \frac{1}{2}AB$. Hence the segments DN, EL and FM can form a triangle half the linear dimensions of triangle ABC.



2. The eight vertices of a cube may be painted black and white, with four of each colour, such that no two vertices of the same colour are adjacent. Label the white vertices with 0s and label the black vertices with 1s. In each move, the 0s become 1s and vice versa. In ten moves, all labels return to their initial values, but not all labels have the same value.



3. Let the lengths of the segments be $a = a_1 \geq a_2 \geq \cdots \geq a_{11}$. Since $1 = a_1 + a_2 + \cdots + a_{11} \leq 11a$, we must have $a \geq \frac{1}{11}$. On the other hand, if $a = \frac{1}{10}$, we may take $a_1 = a_2 = \cdots = a_9 = \frac{1}{10}$ and $a_{10} = a_{11} = \frac{1}{20}$. Then the shortest two segments will not form a triangle with the longest. A larger value of a will only make the longest segment longer, and does not help. Now let $\frac{1}{11} \leq a < \frac{1}{10}$. Then $a_{10} + a_{11} = 1 - (a_1 + a_2 + \cdots + a_9) \geq 1 - \frac{9}{10} = \frac{1}{10} > a_1$. It follows that any three of the segments can form a triangle.

4. The diagram below shows the 15×15 chessboard divided into a central cross of width 3 and four quadrants each a 6×6 square. The numbering shows that all 144 squares in the four quadrants may be visited. If more squares may be visited, then the chess piece must visit one of the squares of the central cross. However, from any such square, the piece can never get to any of the 144 squares in the four quadrants. Even if it can visit all squares in the central cross, the total of 81 is well short of 144.

14	16	18	20	22	24				13	15	17	19	21	23
38	40	42	44	46	48				37	39	41	43	45	47
62	64	66	68	70	72				61	63	65	67	69	71
86	88	90	92	94	96				85	87	89	91	93	95
110	112	114	116	118	120				109	111	113	115	117	119
134	136	138	140	142	144				133	135	137	139	141	143
11	9	7	5	3	1				12	10	8	6	4	2
35	33	31	29	27	25				36	34	32	30	28	26
59	57	55	53	51	49				60	58	56	54	52	50
83	81	79	77	75	73				84	82	80	78	76	74
107	105	103	101	99	97				108	106	104	102	100	98
131	129	127	125	123	121				132	130	128	126	124	122

5. Let the coins be A, B, C, D, E and F. In three weighings, we determine the average weight m of C and E, the average weight n of D and F, and the average weight k of B, E and F. If $m = n = k$, the fake coin is A. If $m = n \neq k$, the fake coin is B. If $m \neq n = k$, the fake coin is C. If $k = m \neq n$, the fake coin is D. If $k \neq m \neq n \neq k$, then the fake coin is E or F. This can be distinguished since $2m + n = 3k$ if it is E, and $m + 2n = 3k$ if it is F.