International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2005.**[1]

1. A palindrome is a positive integer which reads the same from left to right and from right to left. For example, 1, 343 and 2002 are palindromes, while 2005 is not. Is it possible to find 2005 values of n such that both n and $n + 110$ are palindromes?

2. The extensions of the sides AB and DC of a convex quadrilateral $ABCD$ intersect at the point K. M and N are the midpoints of AB and CD, respectively. Prove that if $AD = BC$, then triangle MNK is obtuse.

3. Initially, there is a rook on each of the 64 squares of an 8×8 chessboard. Two rooks attack each other if they are in the same row or column, and there are no other rooks directly in between. In each move, one may take away any rook which attacks an odd number of other rooks still on the chessboard. What is the maximum number of rooks that can be removed?

4. Each side of a polygon is longer than 100 centimetres. Initially, two ants are on the same edge of the polygon, at a distance 10 centimetres from each other. They crawl along the perimeter of the polygon, maintaining the distance of 10 centimetres measured along a straight line.
   (a) Suppose the polygon is convex. Is it always possible for each point on the perimeter of the polygon to be visited by both ants?
   (b) Suppose the polygon is not necessarily convex. Is it always possible for each point on the perimeter of the polygon to be visited by at least one of the ants?

5. Determine the largest positive integer N for which there exist a unique triple (x, y, z) of positive integers such that $99x + 100y + 101z = N$.

6. There are 1000 pots each containing varying amounts of jam, not more than $\frac{1}{100}$-th of the total. Each day, exactly 100 pots are to be chosen, and from each chosen pot, the same amount of jam is eaten. Prove that it is possible to eat up all the jam in a finite number of days.

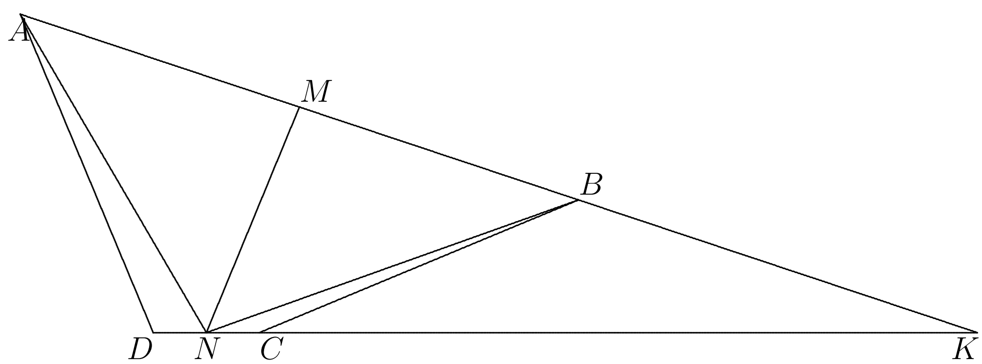
Note: The problems are worth 3, 5, 6, 2+4, 7 and 8 points respectively.

[1]Courtesy of Professor Andy Liu.

Solution to Junior A-Level Fall 2005

1. We can choose $n = 1099\cdots9901$. Clearly, n is a palindrome. Since $n + 110 = 1100\cdots0011$, it is also a palindrome. Since the number of 9s in n is arbitrary, we can certainly find 2005 such values.

2. Suppose $AN < BN$. In triangles MAN and MBN, we have $MA = MB$ and $MN = MN$. Hence $\angle AMN < \angle BMN$. Since $\angle AMN + \angle BMN = 180°$, we have $\angle KMN > 90°$. Similarly, $MD < MC$ implies $\angle KNM > 180°$. Suppose $AN \geq BN$ and $MD \geq MC$. In triangles DAN and CBN, we have $DN = CN$ and $MN = MN$. Hence $\angle ADN \geq \angle BCN$. Similarly, $\angle DAM \geq \angle CBM$. Since $\angle ADN + \angle DAM + \angle BCN + \angle CBM = 360°$, $\angle ADN + \angle DAM \geq 180°$. However, this is a contradiction since $\angle ADN + \angle DAM = 180° - \angle AKD < 180°$.



3. First, note that none of the corner rooks may be removed since each always attacks two other rooks. Moreover, we cannot leave behind only the four corner rooks, as otherwise the last to be taken away will attack two or zero other rooks. We can take away as many as 64-4-1=59 rooks in two stages, as shown in the diagrams below.

•	1	2	3	4	5	6	•
13	∘	∘	∘	∘	∘	∘	7
14	15	16	17	18	19	∘	8
20	21	22	23	24	25	∘	9
26	27	28	29	30	31	∘	10
32	33	34	35	36	37	∘	11
38	39	40	41	42	43	•	12
•	44	45	46	47	48	49	•

•							•
	50	51	52	53	54	55	
						56	
						57	
						58	
						59	
						•	
•							•

4. (a) Let $ABCD$ be a rhombus with BD horizontal and less than 10 centimetres long. Then the segment XY joining the two ants is almost vertical. Let X be the ant initially closer to A and Y be the ant initially closer to C. Then X can never visit C while Y cannot visit A.

   (b) Modify the rhombus $ABCD$ by moving C vertically towards A until AC is less than 10 centimetres long. Then the segment XY joining the two ants is still almost vertical. If XY is initially on AB or AD, neither X nor Y can visit C. If XY is initially on CB or CD, neither X nor Y can visit A.

5. We claim that $N = 5251$ is the largest value. We first show that $99x + 100y + 101z = 5251$ has the unique positive integral solution (50,2,1). Note that $x + y + z = 52$ or 53 since $51 \times 101 < 5251 < 54 \times 99$. Suppose $x + y + z = 52$. Subtracting 100 times this from the given equation, we have $z - x = 51$. We can only have $(x, y, z) = (0, 1, 51)$, but this is not admissible. Suppose $x + y + z = 53$. Subtracting the given equation from 100 times this, we have $x - z = 49$. Neither (49,4,0) nor (51,0,2) is admissible, leaving $(x, y, z) = (50, 2, 1)$ as the unique solution. We next show that for $N = 5251 + k$, $1 \leq k \leq 99$, we have at least two positive integral solutions to $99x+100y+101z = N$. For $1 \leq k \leq 49$, they are $(50-k, k+2, 1)$ and $(51 - k, k, 2)$. For $50 \leq k \leq 97$, they are $(1, 100 - k, k - 48)$ and $(2, 98 - k, k - 47)$. For $k = 98$, they are (52,1,1) and (1,2,50). For $k = 99$, they are (51,2,1) and (1,1,51). Finally, for $N \geq 5351$, modify each of the two solutions for $N - 99$ by adding 1 to x.

6. More generally, let there be m pots each containing varying amounts of jam, not more than $\frac{1}{n}$-th of the total, where $n \leq m$. Each day, exactly n pots are to be chosen, and from each chosen pot, the same amount of jam is eaten. We use induction on n. The basis $n = 1$ is trivial as we can eat up the jam one pot at a time. Suppose there is a strategy to eat all the jam for some $n \geq 1$. We now eat from $n + 1$ pots each day. Let P be the pot with the most jam. We consider two cases:
Case 1. P contains less than $\frac{1}{n+1}$ of the total amount of jam.
Then $m > n + 1$. Choose the $n + 1$ pots containing the least combined amount of jam. Our plan is to eat from each of them the amount equal to what is in the pot with the least amount, so that it becomes empty. However, we must ensure that the condition in the hypothesis still holds afterwards. If P will contain more than $\frac{1}{n+1}$ of the remaining amount of jam, then we modify our plan by reducing the amount eaten from each of the chosen pots, so that P will contain exactly $\frac{1}{n+1}$ of the remaining amount of jam. We then proceed to Case 2. On the other hand, if the condition in the hypothesis holds after our original plan is carried out, we have reduced the number of non-empty pots by one. At some point before or when this number becomes $n + 1$, P will contain exactly $\frac{1}{n+1}$ of the remaining amount of jam, and we proceed to Case 2.
Case 2. P contains exactly $\frac{1}{n+1}$ of the total amount of jam.
First note that each pot other than P contains at most $\frac{1}{n}$ of the total amount of jam not in P. Hence we may apply the strategy when we eat from n pots each day to the pots other than P. At the same time, we also eat from P each day, so that we are eating from $n + 1$ pots as required. By the induction hypothesis, we can eat all the jam from the pots other than P. Since we eat from P each day exactly $\frac{1}{n}$ of what we eat from the other pots, and P starts with exactly $\frac{1}{n}$ of the jam in the other pots initially, we will finish off the jam in P at the same time as we empty the other pots.