International Mathematics
TOURNAMENT OF THE TOWNS

Solution to Senior O-Level Spring 2004[1]

1. Let O be the centre of the circle, K be the point of tangency with BC and H be the point of intersection of AC and BD. Since $AB = BC$, AC is perpendicular to OB by symmetry. Similarly, BD is perpendicular to OC. Since AC intersects BD at H, H is the orthocentre of triangle OBC. Now the radius OK is perpendicular to the tangent BC. Hence the third altitude OK of triangle OBC passes through H.

2. Note that $b = a(10^n + 1)$ so that $\frac{b}{a^2} = \frac{10^n+1}{a}$. Suppose it is an integer d. Since a is an n-digit number, $1 < d < 11$. Since $10^n + 1$ is not divisible by 2, 3 or 5, the only possible value for d is 7. The example $a = 143$ and $b = 143143$ shows that we can indeed have $d = 7$.

3. Let the quadrilateral be $ABCD$ with $AC = 1001$ and $BD = n$. Note that $1002^2 - 1001^2 = 2003$ lies between 44^2 and 45^2. For $45 \leq n \leq 1001$, let M be the common midpoint of AC and BD. Initially, let B lie on AM, so that the degenerate quadrilateral $ABCD$ has perimeter 2002. Now rotate BD about M. When BD is perpendicular to AC, the perimeter of $ABCD$ will exceed 2004. Hence at some point during the rotation, the perimeter of $ABCD$ is exactly 2004. It follows that all values of n between 45 and 1001 inclusive are possible. For $2 \leq n \leq 44$, start with the rhombus $ABCD$ whose perimeter is less than 2004. Translate BD in the direction AC. When C is the midpoint of BD, both AB and AD are longer than 1001, so that the degenerate quadrilateral $ABCD$ has perimeter exceeding $2002 + n \geq 2004$. Hence at some point during the translation, the perimeter of $ABCD$ is exactly 2004. It follows that all values of n between 2 and 44 inclusive are possible. Finally, consider the case $n = 1$. Let M be the point of intersection of AC and BD. Then

$$\begin{aligned} 2004 &= AB + BC + CD + DA \\ &< MA + MB + MB + MC + MC + MD + MD + MA \\ &= 2(AC + BD) \\ &= 2004, \end{aligned}$$

which is a contradiction. It follows that we cannot have $n = 1$.

4. Let the first three terms be $a_1 = a$, $a_2 = a + d$ and $a_3 = a + 2d$, where d is the common difference. Let $a_1^2 = a + kd$, $a_2^2 = a + md$ and $a_3^2 = a + nd$ for some positive integers k, m and n. Then $a^2 = a + kd$, $a^2 + 2ad + d^2 = a + md$ and $a^2 + 4ad + 4d^2 = a + nd$. It follows that $2ad + d^2 = nd - kd$ or $2a + d = m - k$, and $4ad + 4d^2 = nd - kd$ or $4a + 4d = n - k$. Eliminating d, we have $a = \frac{4m-n-3k}{4}$. Hence a is an integral multiple of $\frac{1}{4}$. Eliminating a, we have $d = \frac{n+k-2m}{2}$. Hence d is an integral multiple of $\frac{1}{2}$. Denote by $\{x\}$ the fractional part of x. We consider the following cases.
(1) Let $\{a\} = 0$ and $\{d\} = \frac{1}{2}$. Every term of the progression is an integral multiple of $\frac{1}{2}$ but a_2^2 is not, a contradiction.

[1]Courtesy of Andy Liu.

(2) Let $\{a\} = \frac{1}{2}$. Every term of the progression is an integral multiple of $\frac{1}{2}$ but a_1^2 is not, a contradiction.

(3) Let $\{a\} = \frac{1}{4}$ or $\frac{3}{4}$. Every term of the progression is an integral multiple of $\frac{1}{4}$ but a_1^2 is not, a contradiction.

Thus both a and d are integers, so that every term of the progression is an integer.

5. There are 9×10^9 10-digit numbers. If two of them are non-neighbours, they cannot have the same digits in each of the first nine places. Thus the number of 10-digit numbers we can choose is no more than the number of 9-digit numbers, which is 9×10^8. On the other hand, for each 9-digit number, we can add a unique tenth digit so that the sum of all 10 digits is a multiple of 10. If two of the 10-digit numbers obtained this way differ in only one digit, not both digit sums can be multiples of 10. Hence no two are neighbours among these 9×10^8 10-digit numbers.