International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Spring 2004.**

1. Each day, the price of the shares of the corporation "Soap Bubble, Limited" either increases or decreases by n percent, where n is an integer such that $0 < n < 100$. The price is calculated with unlimited precision. Does there exist an n for which the price can take the same value twice?

2. All angles of a polygonal billiard table have measures in integral numbers of degrees. A tiny billiard ball rolls out of the vertex A of an interior $1°$ angle and moves inside the billiard table, bouncing off its sides according to the law "angle of reflection equals angle of incidence". If the ball passes throigh a vertex, it will drop in and stays there. Prove that the ball will never return to A.

3. The perpendicular projection of a triangular pyramid on some plane has the largest possible area. Prove that this plane is parallel to either a face or two opposite edges of the pyramid.

4. At the beginning of a two-player game, the number 2004! is written on the blackboard. The players move alternately. In each move, a positive integer smaller than the number on the blackboard and divisible by at most 20 different prime numbers is chosen. This is subtracted from the number on the blackboard, which is erased and replaced by the difference. The winner is the player who obtains 0. Does the player who goes first or the one who goes second have a guaranteed win, and how should that be achieved?

5. The parabola $y = x^2$ intersects a circle at exactly two points A and B. If their tangents at A coincide, must their tangents at B also coincide?

6. The audience shuffles a deck of 36 cards, containing 9 cards in each of the suits spades, hearts, diamonds and clubs. A magician predicts the suit of the cards, one at a time, starting with the uppermost one in the face-down deck. The design on the back of each card is an arrow. An assistant examines the deck without changing the order of the cards, and points the arrow on the back each card either towards or away from the magician, according to some system agreed upon in advance with the magician. Is there such a system which enables the magician to guarantee the correct prediction of the suit of at least

   (a) 19 cards;

   (b) 20 cards?

Note: The problems are worth 4, 6, 6, 6, 7 and 3+5points respectively.

Solution to Senior A-Level Spring 2004

1. Suppose the price is repeated after a incrreases and b decreases. Then
$$(100+n)^a(100-n)^b = 100^{a+b}.$$
Now the right side has only 2 and 5 as its prime divisors, so that the same must be true for the left side. The only power of 2 between 101 and 199 is 128. The only number in this range that is 5 times a power of 2 is 160. There are no numbers in this range that are 25 times a power of 2. The only number in this range that is 125 times a power of 2 is 125 itself. Hence $n = 25$, 28 or 60. Then $100 - n = 75$, 72 or 40, so that $n = 60$. However, we cannot have $160^a 40^b = 100^{a+b}$ because the prime factorization of the left side has more 2s than 5s, while that of the right side has an equal number of 2s and 5s.

2. Let a side of the billiard table be parallel to the x-axes and let $\angle ZAB$ be the 1° interior angle. If the ball starts along AB or AZ, it will fall into B or Z respectively. Suppose the ball follows a path $A_0A_1 \dots A_n$ where $A_0 = A = A_n$, with $\angle A_1AB$ and $\angle ZAA_{n-1}$ both strictly between 0° and 1°. Then we cannot have $A_1 = A_{n-1}$ since this can only happen if a segment of the path is parpendicular to a side of the billiard table. Since the former must make an angle of a non-integral number of degrees with the x-axis while the latter makes an angle of an integral number of degrees, the situation is impossible. Now each segment of the path divides the billiard table into two polygons. For the $k-th$ segment $A_{k-1}A_k$, let S_k denote the sum of the interior angles of the polygon to the right side of the directed segment. Consider now the sum $S = S_0 - S_1 + S_2 - \dots + (-1)^n S_n$. Each term is an integral multiple of 180°. Hence S is an even number of degrees. All angles of incidence and angles of reflection cancel in this alternate sum. The interior angles of the billiard table appear in turn, starting with B and ending with Z, and possibly going around several times. Hence each appears the same number of times except for A, which appears one time less. Since we are only interested in the parity of the degrees in S, we can ignore the signs for angles with integral number of degrees. Hence $\angle A_1AB + (-1)^n \angle ZAA_{n-1} + 1^\circ$ is an even number of degrees. This is only possible if n is even and $A_1 = A_{n-1}$. However, this has already been ruled out. Thus the ball can never return to A.

3. If a polygon is orthogonally projected onto a plane, the projection has the largest area when the plane is parallel to the polygon. Let the tetrahedron be $ABCD$. Suppose its orthogonal projection to the plane Π has the largest area. Now the porjection is either a triangle or a convex quadrilateral. In the former case, Π is clearly parallel to a face of the tetrahedron. In the latter case, let the diagonals of the quadrilateral be projected from the sides AC and BD. Perform translations in the direction perpendicular to Π, taking AC into EG and BD into FH, where $EFGH$ is a convex quadrilateral. If Π is parallel to $EFGH$, then it is parallel to both AC and BD. If not, let Ψ be a plane parallel to both AC and BD. As before, perform translations in the direction perpendicular to Ψ, taking AC and BD into coplanar segments. If necessary, perform translations in this plane so that the seqments are the diagonals of a convex quadrilateral $KLMN$. Now the area of the orthogonal projection of $ABCD$ onto Ψ is at least that of $KLMN$, which is equal to the area of $EFGH$, which is in turn greater than that of the orthogonal projection of $ABCD$ onto Π. This is a contradiction.

4. Let P be the product of the first 21 primes. The player who goes second has a winning strategy, by always leaving a multiple of P for the opponent. This guarantees a win since both 2004! and 0 are multiples of P. We first show that the opponent cannot turn the table around. Suppose the first player is left with a multiple n of P. Suppose m is subtracted, leaving a difference d. The only way for d to be a multiple of P is for m to be one also, but this is impossible since m is divisible by at most 20 distinct primes. So the first player must leave behind a difference d which is not a multiple of P. Divide d by P and let the remainder be $r < P$. Since P is the smallest number that is divisible by at least 21 distinct primes, r is divisible by at most 20 distinct primes. Hence the second player can subtract r from d and leave behind another multiple of P.

5. Let the circle be centred at (u, v) with radius e, so that its equation is $(x-u)^2+(y-v)^2 = r^2$. The given geometric information means that the equation $(x-u)^2+(x^2-v)^2 = r^2$ has exactly two real roots, one of which is repeated. Since at least three of the roots are real, all four must be real. If the other root is also repeated, we have tangency at B as well. We must investigate whether the quartic equation could have a triple root and a single root. We may take A to be the point (1,1). Let the x-coordinate of B be $b \neq 1$. Then

$$(x-u)^2 + (x^2-v)^2 - r * 2 = (x-1)^3(x-b).$$

Comparing the coefficients of the cubic terms, we have $3+b = 0$ so that $b = -3$. Compaaring the coefficients of the linear terms, we have $-2u = 8$ so that $u = -4$. Comparing the coefficients of the quadratic terms, we have $1-2v = -6$ so that $v = \frac{7}{2}$. Finally, comparing the constant terms, we have $16 + \frac{49}{4} - r^2 = -3$, so that $r = \frac{5\sqrt{5}}{2}$. In summary, the circle $(x+4)^2 + (y-\frac{7}{2})^2 = (\frac{5\sqrt{5}}{2})^2$ intersects the parabola $y = x^2$ only at $A(1,1)$ and $B(-3,9)$. with common tangents only at A.

6. **First Solution by Che-Yu Liu, Grade 10 student, Taiwan.**

   (a) For each two cards, the assistant can use the backs to give four different signals, corresponding to the suit of the second of the two cards. Thus the magician can "predict" with accuracy the suits of all even-numbered cards, up to card number 34. Now the magician knows the overall composition of the deck, If the last two cards are of the same suit, no further assistance is required. If they are not, the assistant can use the back of card number 35 to indicate whether it is of the higher ranking or lower ranking suit. It follows that the magician can guarantee at least 19 correct answers.

   (b) The assistant examines the odd-numbered cards except card number 1. By the generalized Pigeomhole Principle, at least 5 of them are of the same suit. The assistant uses the backs of cards number 1 and 2 to signal this suit, and the magician always guesses this suit for all odd-numbered cards. The suits of even-numbered cards except card number 2 can be correctly as in (a). Thus the magician can guarantee at least 5+17=22 correct answers even with an arbitrary distribution among the suits.

Second Solution by Andy Lai, Grade 11 student, Taiwan.

(a) The assistant use the back of cards number 1 and 2 to signal the suit of card number 2 or 3, and the magician guesses that suit for both cards number 2 and 3. This is repeated until the magician makes his guesses on cards number 32 and 33. This guarantees 16 correct answers so far since the magician is right at least once on every two consecutive guesses of the same suit. The assistant then uses the backs of cards number 33 and 34 to signal the suit of card number 34. Now the magician knows the overall composition of the deck, If the last two cards are of the same suit, no further assistance is required. If they are not, the assistant can use the back of card number 35 to indicate whether it is of the higher ranking or lower ranking suit. It follows that the magician can guarantee at least 19 correct answers.

(b) Start off as in (a) until the magician guesses the suit of card number 21. At least 10 correct answers have been obtained. Now the assistant uses the backs of cards number 21 and 22 to signal the suit of card 22, and so on until the magician correctly guesses the suit of card number 34. Cards number 35 and 26 are handled as in (a). Thus far, we can again guarantee 19 corrct answers. If either cards number 2 and 3 or cards number 4 and 5 are of the same suit, we have an additional correct answer. Suppose this is not the case. Now the assistant can arrange for the magician to guess correctly card number 2 or 3, and card number 4 or 5. This flexibility is now used to signal the suit of card number 25. For instance, if card number 25 is Spades, then the correct guesses will occur on cards number 2 and 4. If it is Hearts, they occue on cards number 2 and 5; if Diamonds, 3 and 4; and if Clubs, 3 and 5. Similarly, an extra correct answer can come out of cards number 6 to 9 plus 27, of cards number 10 to 13 plus 29, of cards number 14 to 17 plus 31, and of cards number 18 to 21 plus 33. This guarantees at least 24 correct answers.