International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Spring 2004.**

1. The sum of all terms of a finite arithmetical progression of integers is a power of two. Prove that the number of terms is also a power of two.

2. What is the maximal number of checkers that can be placed on an 8×8 checkerboard so that each checker stands on the middle one of three squares in a row diagonally, with exactly one of the other two squares occupied by another checker?

3. Each day, the price of the shares of the corporation "Soap Bubble, Limited" either increases or decreases by n percent, where n is an integer such that $0 < n < 100$. The price is calculated with unlimited precision. Does there exist an n for which the price can take the same value twice?

4. Two circles intersect in points A and B. Their common tangent nearer B touches the circles at points E and F, and intersects the extension of AB at the point M. The point K is chosen on the extention of AM so that $KM = MA$. The line KE intersects the circle containing E again at the point C. The line KF intersects the circle containing F again at the point D. Prove that the points A, C and D are collinear.

5. All sides of a polygonal billiard table are in one of two perpendicular directions. A tiny billiard ball rolls out of the vertex A of an inner $90°$ angle and moves inside the billiard table, bouncing off its sides according to the law "angle of reflection equals angle of incidence". If the ball passes a vertex, it will drop in and srays there. Prove that the ball will never return to A.

6. At the beginning of a two-player game, the number 2004! is written on the blackboard. The players move alternately. In each move, a positive integer smaller than the number on the blackboard and divisible by at most 20 different prime numbers is chosen. This is subtracted from the number on the blackboard, which is erased and replaced by the difference. The winner is the player who obtains 0. Does the player who goes first or the one who goes second have a guaranteed win, and how should that be achieved?

Note: The problems are worth 4, 5, 5, 6, 6 and 7 points respectively.

Solution to Junior A-Level Spring 2004

1. Let the first term of the arithmetic progression be a, the common difference be d and the number of terms be n. Then the sum of all terms is equal to $\frac{1}{2}n(a+a+(n-1)d) = 2^k$ for some positive integer k. Hence $n(2a+(n-1)d) = 2^{k+1}$, and each factor on the left side must be a power of 2.

2. Clearly, no checkers can be placed on any of the 28 outside squares. Moreover, at least one of each set of three squares with the same label must be left vacant. Thus the number of checkers that can be placed is at most 32. If we leave vacant the 28 outside squares and the 4 central squares, it is easy to verify that all conditions are satisfied. Thus the maximum is 32.

			2	1			
		3	1	2	4		
		1	3	4	2		
			4	3			

3. Suppose the price is repeated after a incrreases and b decreases. Then

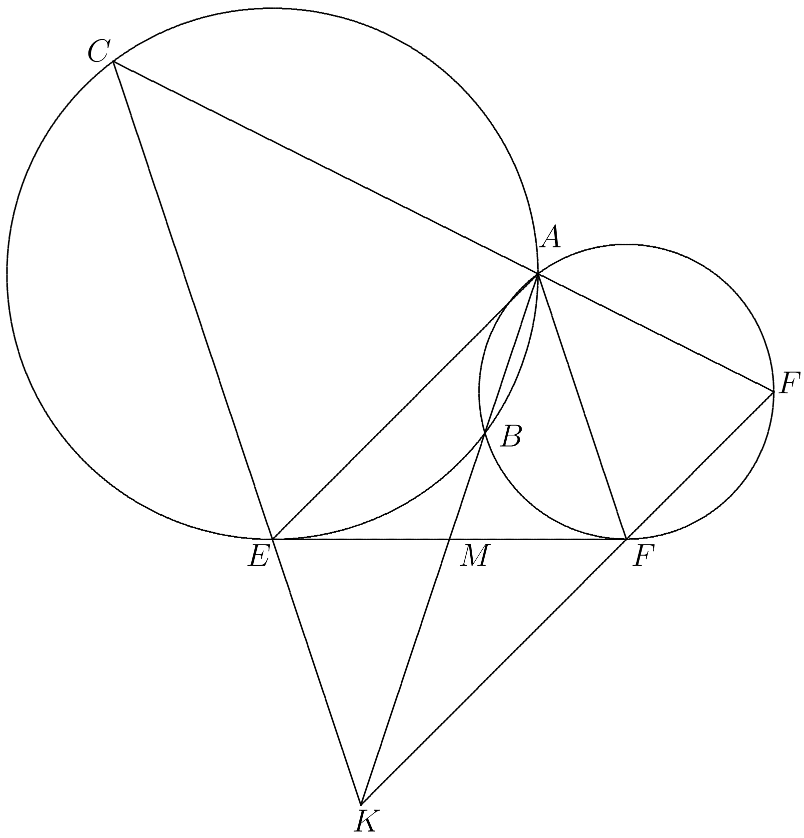
$$(100+n)^a(100-n)^b = 100^{a+b}.$$

Now the right side has only 2 and 5 as its prime divisors, so that the same must be true for the left side. The only power of 2 between 101 and 199 is 128. The only number in this range that is 5 times a power of 2 is 160. There are no numbers in this range that are 25 times a power of 2. The only number in this range that is 125 times a power of 2 is 125 itself. Hence $n = 25$, 28 or 60. Then $100 - n = 75$, 72 or 40, so that $n = 60$. However, we cannot have $160^a 40^b = 100^{a+b}$ because the prime factorization of the left side has more 2s than 5s, while that of the right side has an equal number of 2s and 5s.

4. Since $ME^2 = MA \cdot MB = MF^2$, AK and EF bisect each other, so that $AEKF$ is a parallelogram. Moreover, since EF is tangent to the circles,

$$\angle KCA + \angle KDA + \angle CKD = \angle AEF + \angle AFE + \angle EAF = 180^\circ.$$

It follows that C, A and D rae collinear.



5. Let the sides of the billiard table be parallel to the coordinate axes. If the ball starts along a side, it will fall into the vertex at the other end of the side. Otherwise, the absolute value of the slope of each segment of the ball's path is a positive constant. Hence the ball can only return to A by doubling back along the original path. Thus there must be a point of reversal where the path hits a side in a perpendicular direction. Since the sides are all horizontal or vertical while the path never is, the ball can never return to A.

6. Let P be the product of the first 21 primes. The player who goes second has a winning strategy, by always leaving a multiple of P for the opponent. This guarantees a win since both 2004! and 0 are multiples of P. We first show that the opponent cannot turn the table around. Suppose the first player is left with a multiple n of P. Suppose m is subtracted, leaving a difference d. The only way for d to be a multiple of P is for m to be one also, but this is impossible since m is divisible by at most 20 distinct primes. So the first player must leave behind a difference d which is not a multiple of P. Divide d by P and let the remainder be $r < P$. Since P is the smallest number that is divisible by at least 21 distinct primes, r is divisible by at most 20 distinct primes. Hence the second player can subtract r from d and leave behind another multiple of P.