

**International Mathematics
TOURNAMENT OF THE TOWNS**

Senior O-Level Paper¹

Fall 2004.

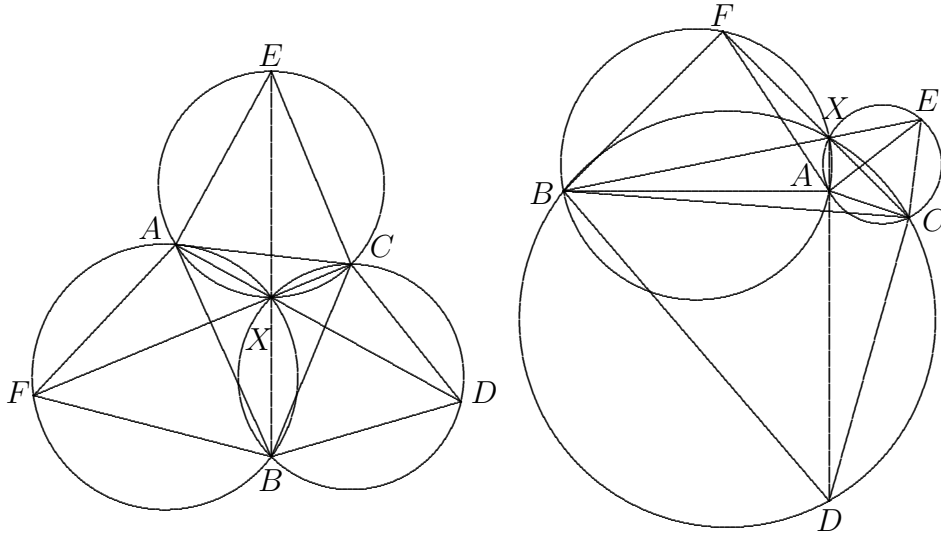
1. Three circles all passing through X intersect one another again pairwise at A , B and C respectively. The extension of the common chord AX of two of the circles intersects the third circle again at D . Similarly, the extensions of BX and CX yield the points E and F respectively. Prove that triangles BCD , CAE and ABF are similar to one another.
2. A bag contains 100 balls, each of which is red, white or blue. If 26 balls are drawn at random, there will always be 10 balls of the same colour among them. What is the smallest number of balls that must be drawn, at random, in order to guarantee that there will be 30 balls of the same colour among them?
3. $P(x)$ and $Q(x)$ are non-constant polynomials such that for all x , $P(P(x)) = Q(Q(x))$ and $P(P(P(x))) = Q(Q(Q(x)))$. Is it necessarily true that $P(x) = Q(x)$ for all x ?
4. In how many ways can 2004 be expressed as the sum of one or more positive integers in non-decreasing order, such that the difference between the last term and the first term is at most 1?
5. For which positive integers n is it possible to arrange the numbers from 1 to n in some order, such that the average of any group of two or more adjacent numbers is not an integer?

Note: The problems are worth 3, 3, 4, 4 and 5 points respectively.

¹Courtesy of Andy Liu.

Solution to Senior O-Level Fall 2004

1. We have $\angle EAC = \angle EXC = \angle FXB = \angle FAB$. Denote the common value by α . Similarly, we have $\angle FBA = \angle DXC = \angle DBC = \beta$ and $\angle DCB = \angle DXB = \angle ECA = \gamma$. Note that $\alpha + \beta + \gamma = \angle FXB + \angle DXC + \angle DXB = 180^\circ$. Hence $\angle BDC = 180^\circ - \beta - \gamma = \alpha$, $\angle CEA = 180^\circ - \gamma - \alpha = \beta$ and $\angle AFB = 180^\circ - \alpha - \beta = \gamma$. It follows that triangles DBC , AEC and ABF are indeed similar to one another.



Note that X may lie on the extension of one of the common chords, say DA . We have $\angle EAC = \angle EXC = \angle FXB = \angle FAB$. Denote the common value by α . Similarly, we have $\angle CEA = \angle DXC = \angle DBC = \beta$ and $\angle AFB = \angle DXB = \angle DCB = \gamma$. As before, we have $\alpha + \beta + \gamma = \angle FXB + \angle DXC + \angle DXB = 180^\circ$. Hence $\angle BDC = 180^\circ - \beta - \gamma = \alpha$, $\angle CEA = 180^\circ - \gamma - \alpha = \beta$ and $\angle FAB = 180^\circ - \alpha - \beta = \gamma$. It follows that triangles DBC , AEC and ABF are indeed similar to one another.

2. We first show that 65 is not enough. We may have in the bag 47 red, 7 white and 46 blue balls. If 26 are drawn at random, the number of red and blue balls is at least 19. By the Mean Value Principle, there are either at least 10 red balls or at least 10 blue balls, so that the requirement is satisfied. Now if we draw only 65 balls, we may end up with 29 red, 7 white and 29 blue balls. We now show that 66 is enough. We may assume that the number of white balls is not more than the number of red balls and not more than the number of blue balls. If there are at most 7 white balls, then among the 66 balls drawn, the number of red or blue balls is at least 59, so that the desired result follows from the Pigeonhole Principle. If there are at least 9 white balls, then we may draw 9 red, 8 white and 9 blue balls for a total of 26 balls without 10 of the same colour. Hence the number of white balls must be 8. Since we cannot have at least 9 red and at least 9 blue balls in the bag, we may assume that there are exactly 8 blue balls. When we draw 66 balls, we will get at least 50 red balls.
3. Since $P(x)$ is a polynomial, so is $P(P(x))$, and it takes on infinitely many values. Let x be any of these values. Then $x = P(P(t))$ for some t . Hence $P(x) = P(P(P(t))) = Q(Q(Q(t))) = Q(x)$. Since $Q(x)$ is also a polynomial, and it agrees with $P(x)$ on infinitely many values, we must have $P(x) = Q(x)$ for all x .

4. Consider any k where $1 \leq k \leq 2004$. Use the Division Algorithm to determine the unique pair of integers (q, r) such that $2004 = kq + r$ with $0 \leq r \leq k - 1$. Then r copies of $q + 1$ and $k - r$ copies of q will add up to 2004. Thus there is one desired expression for each value of k , which is clearly unique. Hence there are 2004 such expressions in all.
5. The sum of n consecutive numbers is $\frac{n(2a+n-1)}{2}$ where a is the first of these numbers. Their average is $\frac{2a+n-1}{2}$, which is an integer if and only if n is odd. In our problem, n cannot be odd. We now show that n can be any even number. Arrange the n numbers in their natural order and group them into pairs. Reverse the order within each pair to yield the arrangement $2, 1, 4, 3, 6, 5, \dots, n, n - 1$. Consider any k where $2 \leq k \leq n$. Consider first the case where k is odd. Any k adjacent numbers in our arrangement consist of k consecutive integers except that the one which is not in a pair is replaced by its partner, which differs from it by 1. Thus the sum of these k numbers is $mk \pm 1$ for some m , so that their average is not an integer. Finally, consider the case where k is even. Any k adjacent numbers in our arrangement consist of k consecutive integers, possibly with the two at the ends not being in pairs and replaced by their partners. Since one would be increased by 1 while the other would be decreased by 1, the sum is not affected by the replacement. So the average is not an integer.