International Mathematics

TOURNAMENT OF THE TOWNS

Senior A-Level Paper[1] **Fall 2004.**

1. The functions f and g are such that $g(f(x)) = x$ and $f(g(y)) = y$ for any real numbers x and y. If for all real numbers x, $f(x) = kx + h(x)$ for some constant k and some periodic function $h(x)$, prove that $g(x)$ can similarly be expressed as a sum of a linear function and a periodic function. A function h is said to be periodic if for any real number x, $h(x + p) = h(x)$ for some fixed real number p.

2. Two players alternately remove pebbles from a pile. In each move, the first player must remove either 1 or 10 pebbles, while the second player must remove either m or n pebbles. Whoever cannot make a move loses. If the first player can guarantee a win regardless of the initial number of pebbles in the pile, determine m and n.

3. On a blackboard are written four numbers. They are the values, in some order, of $x + y$, $x - y$, xy and $\frac{x}{y}$ where x and y are positive numbers. Prove that x and y are uniquely determined.

4. A circle with centre I is inside another circle with centre O. AB is a variable chord of the larger circle which is tangent to the smaller circle. Determine the locus of the circumcentre of triangle IAB.

5. We have many copies of each of two rectangles. If a rectangle similar to the first can be made by putting together copies of the second, prove that a rectangle similar to the second can be made by putting together copies of the first, with no overlapping in both instances.

6. Let $n \geq 5$ be a fixed odd prime number. A triangle is said to be admissible if the measure of each of its angles is of the form $\frac{m}{n}180°$ for some positive integer m. Initially, there is one admissible triangle on the table. In each move, one may pick up a triangle from the table and cut it into two admissible ones, neither of which is similar to any other triangle on the table. The two new triangles are put back on the table. After a while, no more moves can be made. Prove that at that point, every admissible triangle is similar to some triangle on the table.

7. From a point O are four rays OA, OC, OB and OD in that order, such that $\angle AOB = \angle COD$. A circle tangent to OA and OB intersects a circle tangent to OC and OD at E and F. Prove that $\angle AOE = \angle DOF$.

Note: The problems are worth 5, 5, 5, 6, 7, 8 and 8 points respectively.

[1] Courtesy of Andy Liu.

Solution to Senior A-Level Fall 2004

1. Let $y = f(x) = kx + h(x)$. Then $y + kp = k(x + p) + h(x + p) = f(x + p)$. It follows that $g(y + kp) = x + p = g(y) + p$. Let $\ell(y) = g(y) - \frac{y}{k}$. Then

$$\ell(y + kp) = g(y + kp) - \frac{y + kp}{k} = g(y) + p - \frac{y}{k} - p = \ell(y).$$

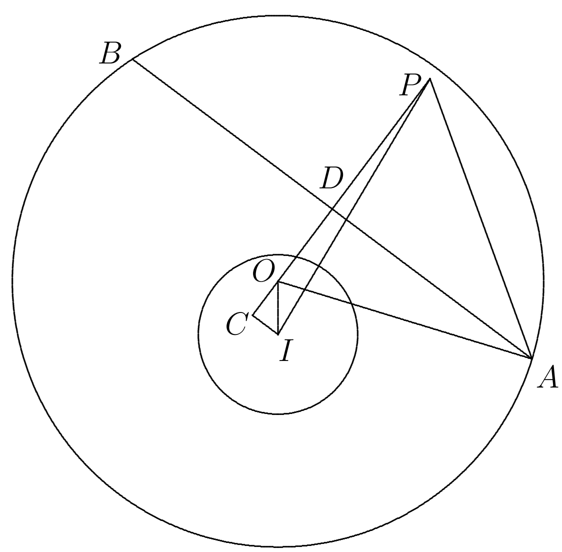
Hence $\ell(y)$ is a periodic function, and $g(y) = \frac{y}{k} + \ell(y)$.

2. Let the first player be Alexei, the second player be Boris, and the total number of pebbles be t. We may assume that $m \leq n$. Suppose $m \leq 8$. If $t = m + 1$, then Alexei can take only 1 pebble and Boris wins by taking the rest. Suppose $n = m + 9$. If $t = m + 10$, then whether Alexei takes 1 or 10 pebbles, Boris can still take the rest and wins. Suppose $m \geq 9$ and $n \neq m+9$. If $t \leq m$, then Alexei wins by taking 1 pebble, leaving Boris with no response. Suppose $t > m$. Alexei has two moves, one of which does not leave behind m pebbles and one of which does not leave behind n pebbles. Suppose taking 1 pebble leaves behind n and taking 10 pebbles leaves behind m. This would mean $n = m+9$, which is not the case. Hence Alexei has a move which leaves behind neither m nor n pebbles, so that the game continues. Since the game cannot continue forever, Boris must eventually lose.

3. Note that $(x+y)+(x-y) = 2x$ while $(xy)(\frac{x}{y}) = x^2$, and that only $x-y$ can be non-positive. We consider three cases.
Case 1. All four numbers are positive.
Let a, b, c and d denote $x+y$, $x-y$, xy and $\frac{x}{y}$ in some order. Choose a pair of them and check if the square of their sum is four times the product of the other two numbers. The pair can be chosen in six ways. There are three subcases.
Subcase 1a. This is satisfied by two disjoint pairs.
We may assume that we have $(a+b)^2 = 4cd$ and $(c+d)^2 = 4ab$. Adding these two equations yields $(a-b)^2 + (c-d)^2 = 0$ so that $a = b$ and $c = d$. Substituting back into $(a+b)^2 = 4cd$, we have $a = \pm c$. Since all four numbers are positive, we must have $a = b = c = d$. This is a contradiction since $x + y \neq x - y$.
Subcase 1b. This is satisfied by two intersecting pairs.
We may assume that we have $(a+b)^2 = 4cd$ and $(a+c)^2 = 4bd$ with $b \neq c$. Then we have $b(a+b)^2 = 4bcd = c(a+c)^2$, or equivalently $(b-c)(a^2 + 2a(b+c) + (b^2+bc+c^2)) = 0$. This is a contradiction since $b - c \neq 0$ while $a^2 + 2a(b+c)_{(}b^2+bc+c^2) > 0$.
Subcase 1c. This is satisfied by only one pair.
We may assume that $(a+b)^2 = 4cd$. Then we know that the larger one of a and b is $x+y$ and the smaller one $x-y$. We can determine x and y uniquely.
Case 2. One of the numbers is 0. We know that $x = y$ so that $\frac{x}{y} = 1$ must also be among the four numbers. The other two are $x+y = 2x$ and $xy = x^2$. Since their product is $2x^3$, we can determine $x = y$ uniquely.
Case 3. One of the numbers is negative.
We know that $x < y$ and $\frac{x}{y} < 1$. Check how many numbers in $S = \{x+y, xy, \frac{x}{y}\}$ lie strictly between 0 and 1. There are three subcases.

Subcase 3a. There is exactly one such number.
We know that this number is $\frac{x}{y}$, and we can determine x and y uniquely from $x-y$ and $\frac{x}{y}$.
Subcase 3b. There are exactly two such numbers.
We cannot have $x+y<1$. Otherwise, we must have $x<1$ and $y<1$ so that $xy<1$, but then all three numbers in S lie strictly between 0 and 1. Hence $x+y>1$ is the largest number in S, and we can determine x and y uniquely from $x-y$ and $x+y$.
Subcase 3c. There are exactly three such numbers.
From $x+y<1$, we have $x<1$ and $y<1$ so that $xy<x+y$ and $xy<\frac{x}{y}$. Hence the smallest number in S is xy, and we can determine x and y uniquely from $x-y$ and xy.

4. The circumcentre P of triangle IAB lies on the line through O perpendicular to AB. Let this line cut AB at D, and let C be the point on this line such that CI is perpendicular to it. Let d denote the distance OI, r the radius of the circle with centre I, and R the radius of the circle with centre O. Then $PA^2=PD^2+AD^2=PD^2+R^2-OD^2$ and $PI^2=CI^2+PC^2=d^2-(r-OD)^2+(PD+r)^2$. These two expressions are equal to each other since $PA=PI$. Simplification yields $R^2-d^2=2r(OD+PD)=2rPO$. Hence $PO=\frac{R^2-d^2}{2r}$ is a fixed distance, so that the locus of P is a circle with this radius and centre O.

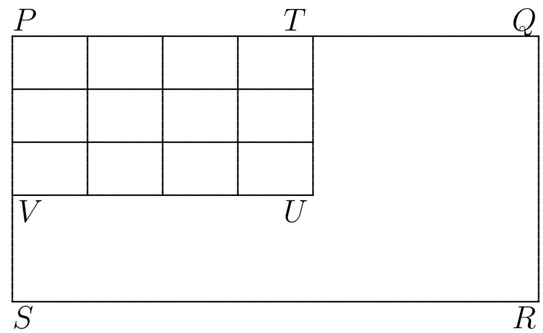


5. Suppose we have an $a_1\times a_2$ rectangle A and a $b_1\times b_2$ rectangle B. Any rectangle $PQRS$ that can be constructed from copies of A has dimensions $(u_1a_1+u_2a_2)\times(v_1a_1+v_2a_2)$ for some non-negative integers $u_1,\ u_2, v_1$ and v_2. If $PQRS$ is similar to B, then

$$\frac{b_1}{b_2}=\frac{u_1a_1+u_2a_2}{v_1a_1+v_2a_2}.$$

We first consider the case where $\frac{a_1}{a_2}$ is rational, so that it is equal to $\frac{m_1}{m_2}$ for some positive integers m_1 and m_2. Then $\frac{b_1}{b_2}=\frac{u_1m_1+u_2m_2}{v_1m_1+v_2m_2}=\frac{n_1}{n_2}$ for some positive integers n_1 and n_2, so that it is also rational. Using n_1n_2 copies of B, we can construct a square of side $s=n_2b_1+n_1b_2$. Using m_1m_2 copies of this square, we can construct an $sm_1\times sm_2$ rectangle which is similar to A.

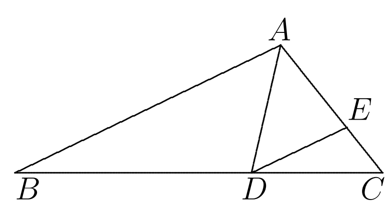
We now consider the case where $\frac{a_1}{a_2}$ is irrational. We claim that in constructing the rectangle $PQRS$ with copies of A, all the copies must be in the same orientation. Let $PTUV$ be the largest subrectangle of $PQRS$ that can be constructed with copies of A all in the same orientation. Suppose U is in the interior of $PQRS$, as illustrated in the diagram below.



If the line TU can be extended without cutting in interior of a copy of A, then the space immediately below UV must be filled with copies of A in the same orientation as those above, as otherwise it contradicts the irrationality of $\frac{a_1}{a_2}$. However, now it contradicts the maximality of $PTUV$. Hence TU cannot be so extended, but this implies that VU can, and we have a contradiction as well. It follows that U must lie on QR or RS. We may assume by symmetry that it lies on QR, so that T coincides with Q. However, the space immediately below UV must be filled with copies of A in the same orientation as those above. This contradicts the maximality of $PTUV$ unless U coincides with R and V with S. Thus our claim is justified. Suppose this construction uses k_1k_2 copies of A in k_1 rows and k_2 columns for some positive integers k_1 and k_2. Then $\frac{k_1a_1}{k_2a_2} = \frac{b_1}{b_2}$ so that $\frac{k_2b_1}{k_1b_2} = \frac{a_1}{a_2}$. Hence we can construct a rectangle similar to A using k_1k_2 copies of B in k_2 rows and k_1 columns.

6. Let the measures of the angles of a resolvable triangle be $\frac{a}{n}$, $\frac{b}{n}$ and $\frac{c}{n}$ times 180°, where a, b and c are positive integers such that $a + b + c = n$. We label such a triangle (a, b, c). For $n = 3$, there is only one resolvable triangle, namely (1,1,1), and the result is trivially true. For $n = 5$, we have (3,1,1) and (2,2,1). Each can be cut into two triangles which are similar to itself and to the other. Thus the result is also true. Henceforth, we assume that $n \geq 7$.

   We generalize the case $n = 5$ as follows. We claim that whenever a resolvable triangle T can be cut into two resolvable ones, one similar to itself and another similar to a different resolvable triangle S, then S can also be cut into two such triangles.



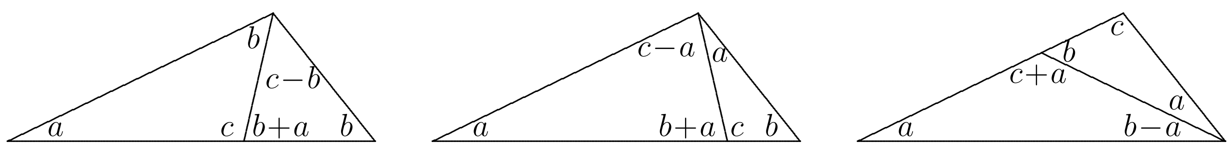
   Let ABC be a resolvable triangle which is cut into two resolvable ones DBA and DCA, with DBA similar to ABC. Then $\angle BAD = \angle BCA$. Now cut DCA along DE parallel to BA. Clearly, EDC is similar to ABC. Since $\angle EDA = \angle BAD = \angle BCA$, EDA is also similar to DCA. This justifies the claim.

   Two such triangles are said to be *compatible* with each other, and the dissection dividing either into triangles similar to both is called their common dissection.

For a fixed n, construct a graph as follows. Each vertex represents a similarity type of resolvable triangles. Two vertices are joined by an edge if and only if the triangles they represent are compatible with each other. Colour red the vertex representing the resolvable triangle given initially, and any other vertices as the triangles they represent appear on the table. We shall only use a common dissection to cut a resolvable triangle into a compatible pair. It follows that once coloured red, a vertex remains red.

Suppose not all vertices are red. If the graph is connected, then there is a pair of adjacent vertices exactly one of which is red. We can make the other vertex red by performing a common dissection. Hence the desired result follows if we can prove that the graph is indeed connected.

The degree of each vertex representing a non-isosceles resolvable triangle is 3. This is because there are common dissections with three other triangles. If the triangle is (a, b, c) where $a < b < c$, then it has a common dissection with each of $(c - b, b, b + a)$, $(b - a, a, c + a)$ and $(c - a, a, b + a)$.



The degree of each vertex representing an isosceles resolable triangle is 1. If it is of the form (a, b, b) where $a < b$, we can only the second or the third common dissection to generate $(b - a, a, b + a)$. If it is of the form (a, c, c) where $a < c$, we can use either the first or the second common dissection to generate $(a, 2a, c - a)$. Moreover, the newly generated triangle can only be isosceles if $n = 5$. Since we are now concerned only with the cases $n \geq 7$, we can safely removal such vertices without affecting the connectivity of the graph. Of course, some of the other vertices will have their degrees reduced from 3 to 2.

Let (a, b, c) be a resolvable triangle with $a \leq b \leq c$. Put the vertex representing it in level a. The vertices on level 1 form a chain $(1,2,n - 3)$ — $(1,3,n - 4)$ — $\cdots$ — $(1, \frac{n-3}{2}, \frac{n+1}{2})$ by the third common dissection. We claim that each vertex in level $a > 1$ is either joined to some vertex at a lower level, either directly or via a chain in level a. Then we can conclude that the graph is connected.

If $a + b > c > b$, we use the first common dissection to obtain $(b + a, b, c - b)$. Since $c - b < a$, the vertex representing this triangle is in a lower level. If $2a > b > a$, we use the third common dissection to obtain $(b - a, a, c + a)$. Since $b - a < a$, the vertex representing this triangle is in a lower level.

Suppose $c > a + b \geq 3a$. We may use the second common dissection to obtain $(a, b + a, c - a)$. For some positive integer k, we will have $a + (b + ka) > c - ka$. Alternatively, we may use the third common division to obtain $(a, b - a, c + a)$. For some positive integer ℓ, we will have $2a > b - \ell a$. In both cases, we are moving within the same level towards a vertex which allows for descent into a lower level.

We will have a problem in the first approach if $b + ka = c - ka$, and in the second approach if $2a = b - \ell a$. Either may occur, but if they occur simultaneously, we have $b = (\ell + 1)a$ while $c = (2k + \ell + 1)a$. Since n is prime, this is only possible if $a = 1$. However, we have already proved that level 1 is connected.

7. Let the circles inscribed in $\angle AOB$ and $\angle COD$ have centres P and Q, and tangent to OA and OD at K and L, respectively. Then we have $\angle POK = \frac{1}{2}\angle AOB = \frac{1}{2}\angle COD = \angle QOL$ and $\angle PKO = 90° = \angle QLO$. Hence triangles POK and QOL are similar. It follows that $\frac{PO}{QO} = \frac{PK}{QL} = \frac{PE}{QE} = \frac{PF}{QF}$, so that the circumcircle of triangle OEF is the locus of all points M satisfying $\frac{PM}{QM} = \frac{PK}{QL}$. Now PQ will intersect this circle at the midpoint I of the arc EF. Hence $\angle IOE = \angle IOF$. Moreover, since $\frac{PI}{QI} = \frac{PO}{QO}$, we have $\angle POI = \angle QOI$. Hence $\angle AOE = \angle AOP + \angle POI - \angle IOE = \angle DOQ + \angle QOI - \angle IOF = \angle DOF$.