International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper[1] **Fall 2004.**

1. Is it possible to arrange the numbers from 1 to 2004 inclusive in some order such that the sum of any ten adjacent numbers is divisible by 10?

2. A bag contains 111 balls, each of which is green, red, white or blue. If 100 balls are drawn at random, there will always be 4 balls of different colours among them. What is the smallest number of balls that must be drawn, at random, in order to guarantee that there will be 3 balls of different colours among them?

3. Various pairs of towns in Russia were linked by direct bus services with no intermediate stops. Alexei Frugal bought one ticket for each route, which allowed travel in either direction but not returning on the same route. He started from Moscow, used up all his tickets without buying any new ones, and finished at Kaliningrad. Boris Lavish bought n tickets for each route, and started from Moscow. However, after using some of his tickets, he got stuck in some town which he could not leave without buying a new ticket. Prove that he got stuck in either Moscow or Kaliningrad.

4. Given a line and a circle which do not intersect, use straight edge and compass to construct a square with two adjacent vertices on the line and the other two on the circle, assuming that such a square exists.

5. In how many ways can 2004 be expressed as the sum of one or more positive integers in non-decreasing order, such that the difference between the last term and the first term is at most 1?

Note: The problems are worth 3, 4, 4, 5 and 5 points respectively.

[1] Courtesy of Andy Liu.

Solution to Junior O-Level Fall 2004

1. We may replace each number by its units digit since this has no effect on divisibility by 10. Then we have 200 of each of 5, 6, 7, 8, 9 and 0, but 201 of each of 1, 2, 3 and 4. Suppose the desired arrangement is possible. Then the 11-th digit must be identical to the 1-st one, the 12-th to the 2-nd, and so on, forming a sequence of period 10. However, $0+1+2+\cdots+9=45$ is not divisible by 10. Hence the period must contain some repeated digit. This digit would have to appear at least 400 times, but none appears more than 201 times. Thus the task is impossible.

2. We first show that 87 is not enough. We may have in the bag 75 green, 12 red, 12 white and 12 blue balls. The total number of balls of any three colours is at most 99. If 100 are drawn at random, there will be 4 balls of different colours. Hence the requirement is satisfied. Now if we draw only 87 balls, we may end up with 75 green and 12 white balls. We now show that 88 is enough. By symmetry, we may assume that the numbers of green, red, white and blue balls is non-increasing. We must have at least 12 blue balls as otherwise we may not have a blue one when we draw 100 balls. Hence there are at least 24 white and blue balls, meaning that the total number of balls of any two colours is at most $111-24=87$. The desired result follows immediately.

3. Consider any town other than Moscow and Kaliningrad. Suppose Alexei visited it k times. Then he came in using k tickets and went out using another k tickets. Hence the number of his tickets with this town on them was $2k$. The number of Boris' tickets with this town on them was $2kn$. They allowed Boris to enter and depart kn times, after which Boris could not come back and be stuck there.

4. Let the given line be horizontal and the circle be above it. Draw the vertical diameter of the circle, and extend it to cut the line at O. From O, draw two lines of slopes ± 2. Suppose they are tangent to the circle at Q and R respectively. Drop perpendiculars from Q and R onto the given line at P and S respectively. Then $PQRS$ is the desired square. If each of the two lines cut the circle at two points, take either the closer pair or the farther pair as Q and R and repeat as before. If the two lines miss the circle completely, the square will not exist, but this is given not to be the case.

5. Consider any k where $1 \leq k \leq 2004$. Use the Division Algorithm to determine the unique pair of integers (q, r) such that $2004 = kq + r$ with $0 \leq r \leq k-1$. Then r copies of $q+1$ and $k-r$ copies of q will add up to 2004. Thus there is one desired expression for each value of k, which is clearly unique. Hence there are 2004 such expressions in all.