Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

O-Level Paper **Spring 2003.**

1 Let S be an entire amount of money (\$2003),

a_i be amount of money in i-pocket, $i = 1, 2, \ldots, M$. Then

$$a_i < N, \qquad S = \sum_{i=1}^{M} a_i < MN. \tag{1}$$

Let us assume that each purse contains no less than M dollars in it. Let b_i be amount of money in i-purse. Then

$$b_i \geq M, \qquad S = \sum_{i=1}^{N} b_i \geq MN. \tag{2}$$

Contradiction.

2 Yes, it could happen.

Example. Consider a 100-gon with sides:

$$1, 1, 2, 2^2, \ldots, 2^{98}, 2^{99} - 1.$$

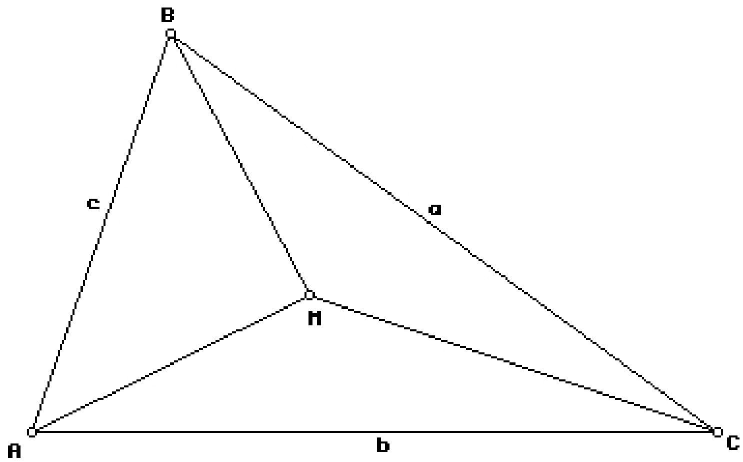
Since $1 + 1 + 2 + \ldots + 2^{98} = 2^{99} > 2^{99} - 1$ it is possible to construct 100-gon with these sides. On the other hand, one cannot construct a polygon from any lesser number of sides. Really, consider two cases:

(a) Side $(2^{99} - 1)$ is among selected.
Then even if the shortest side is absent, $1 + 2 + \ldots + 2^{98} = 2^{99} - 1$.

(b) The longest selected side is 2^k, $1 \leq k \leq 2^{98}$.
Then $1 + 1 + \ldots + 2^{k-1} = 2^k$.

3 Let $\angle AMC = \beta$, $\angle BMC = \alpha$, $\angle AMB = \gamma$, $AC = b$, $BC = a$, $AB = c$, R, r_1, r_2 and r_3 be the radii of the circumcircles of $\triangle ABC$, $\triangle AMC$, $\triangle BMC$ and $\triangle BMA$ respectively. Then formulae $b = 2R \sin \angle B$, $b = 2r_1 \sin \beta$ and condition $r_1 \geq R$ imply that $\sin \beta \leq \sin B$. Similarly, $\sin \alpha \leq \sin A$, $\sin \gamma \leq \sin C$.

Note that $\beta > B$, $\alpha > A$, $\gamma > C$.



Consider two cases:

(a) $\triangle ABC$ is acute.

Then $\beta > B$ and $\sin \beta \leq \sin B$ imply that $\beta \geq \pi - B$. Similarly, $\alpha \geq \pi - A$, $\gamma \geq \pi - C$. Then

$$2\pi = \alpha + \beta + \gamma \geq 3\pi - A - B - C = 2\pi$$

and therefore $\beta = \pi - B$, $\alpha = \pi - A$, $\gamma = \pi - C$ which imply $r_i = R$.

(b) $\triangle ABC$ is not acute.

Assume that $B \geq \frac{\pi}{2}$. Then $\beta > \frac{\pi}{2}$ and

$$2\pi = \alpha + \beta + \gamma > \frac{5\pi}{2} - A - C = \frac{3\pi}{2} + B.$$

Then $B < \frac{\pi}{2}$. Contradiction. This case is impossible.

4 The answer is 50.

Let b_k be a rearranged sequence. Note, that the given operation changes a parity of the next term. I.e., if sum of the digits of b_k is odd/even, then sum of the digits of b_{k+1} is even/odd respectively.

Let us assume that both b_k and b_{k+10} remain on their original places. Note, that the parities of b_k and b_{k+10} are always different. On the other hand, to get b_{k+10} from b_k, one need to change parity an even number of times; so the parities in question should be the same. This implies that a maximal number of terms which could remain on their places does not exceed 50.

Example, in which 50 is achieved:

$$00 \nearrow 09, 19 \searrow 10, 20 \nearrow 29, 39 \searrow 30, 40 \nearrow 49, 59 \searrow 50, 60 \nearrow 69, 79 \searrow 70, 80 \nearrow 89, 99 \searrow 90$$

5 Note that $\frac{1}{2}b < a < b$ implies $a < \sqrt{ab} < b$. Let us choose point E on BC such that $AE = \sqrt{ab}$. It is possible due to inequality $BE = \sqrt{ab - a^2} < b$.

Let F be a point of intersection of AE and $DF \perp AE$. Calculating the area of $\triangle AED$ in two ways we get $\frac{1}{2}AE \cdot DF = \frac{1}{2}AD \cdot CD$. Then $FD = ab/\sqrt{ab} = \sqrt{ab}$.

Since $AF = \sqrt{b^2 - ab} < \sqrt{ab} = AE$ (due to inequality $b < 2a$) point F belongs to AE.

Now $\triangle ABE$, $\triangle AFD$ and quadrilateral $DFEC$ could be rearranged into a square by parallel translation of $\triangle ABE$ into $\triangle DCM$ and $\triangle ADF$ into $\triangle EMK$. One can justify it.