Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper **Spring 2003.**

1 **Solution 1.** The longest edge of the pyramid is a chord of the circumscribed sphere and thus it does not exceed diameter of the sphere: $a \le 2R$. Projection of insphere onto the shortest altitude of the pyramid is strictly contained in the projection of the pyramid onto this altitude. So, $2r < h$. Multiplying inequalities we get $2r \cdot a < h \cdot 2R$, which is equivalent to $\dfrac{a}{h} < \dfrac{R}{r}$.

Solution 2. Let us calculate the volume of the pyramid in two ways: $V = \frac{1}{3}H_j S_j$ and $V = \frac{1}{3}r(S_1 + S_2 + S_3 + S_4)$, where S_j is the area of j-th face, and H_j is a corresponding altitude. Thus $H_j = 3V/S_j$ and $h = 3V/S_{\max}$, where $S_{\max} = \max_j S_j$ is the area of the face with the largest area.

Therefore, $r = 3V/(S_1+S_2+S_3+S_4)$. Note that $(S_1+S_2+S_3+S_4) > 2S_{\max}$. Really, if we project the pyramid onto one of its faces (treated as a base) then a projections of the lateral faces will cover the base. Since area of projection is less than the area of the face itself (because none of lateral faces is parallel to the base) we get our inequality.

Then
$$\frac{R}{r} = \frac{R(S_1+S_2+S_3+S_4)}{3V} > \frac{2RS_{\max}}{3V} = \frac{2R}{h} \ge \frac{a}{h}.$$

2 Answer: $\deg P = 1$.

Solution. We consider a more general problem when a_i are integers (not necessarily positive).

(i) $\deg P = 0$ then $P = c = \text{const}$ and all $a_i = P(a_{i+1})$ are equal which contradicts conditions.

(ii) $\deg P = 1$ is possible: for example, $a_i = i$, $P(x) = x - 1$.

(iii) $m = \deg P \geq 2$. Let us prove that such sequence $\{a_i\}$ does not exists.

Lemma. *If $m \geq 2$ then there exists a constant C such that $\forall x : |x| \geq C \quad |P(x)| > |x|$.*

Proof. Let $P(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$ with $b_m \neq 0$. Then for $|x| \geq 1$

$$|P(x)| \geq |b_m| \cdot |x|^m - \Big(|b_{m-1}| + |b_{m-2} + \cdots + |b_0|\Big)\bigg)|x|^{m-1} \geq$$
$$|x|^{m-1}\bigg(|b_m| \cdot |x| - \Big(|b_{m-1}| + |b_{m-2} + \cdots + |b_0|\Big)\bigg)$$

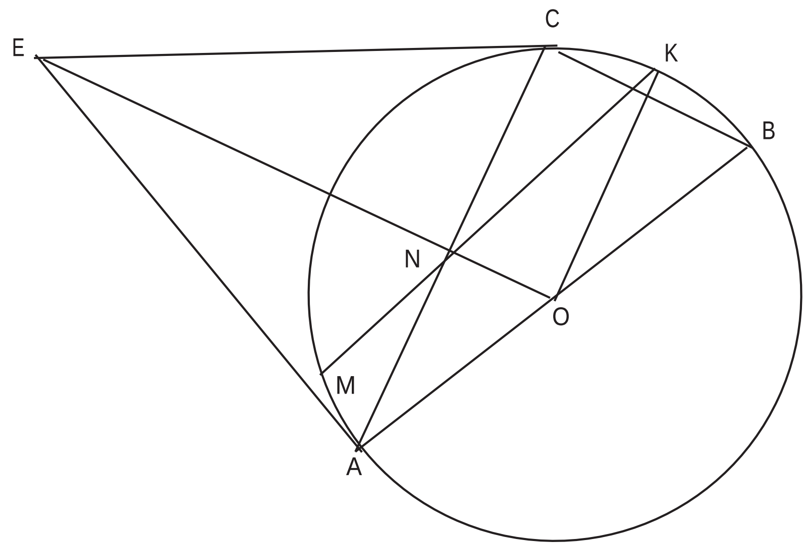
which is larger than $|x|^{m-1}$ as $|x| \geq \big(|b_{m-1}| + |b_{m-2} + \cdots + |b_0| + 1\big)/|b_m|$ and in turn $|x|^{m-1} \geq |x|$. □

Since a_i are distinct integers, for any C there exists M such that $\forall i \geq M \quad |a_i| \geq C$. Then according to Lemma, for $i \geq M \quad |a_i| = |P(a_{i+1})| \geq |a_{i+1}|$ and therefore $|a_i|$ are bounded. Contradiction.

3 First let us notice that no vertex can be covered by an interior of a triangle. So, it should be covered by edges. Note that if an interior of edge covers a vertex, the sum of adjacent angles covered by triangle is exactly 180°. At the same time the sum of angles adjacent to vertex of cube is 270°. Therefore, at least 90° at each vertex should be covered by angles of triangles. So angles of triangles cover at least $8 \cdot 90°$ and there should be at least $8 \cdot 90°/180° = 4$ of triangles.

Consider T-shaped envelope of a cube, consisting of two rectangles. Each of them can be covered by 2 triangles. So, it is possible to cover a cube by 4 triangles.

4



Let O be a center of the circle. Since $\triangle ABC$ is a right triangle, O is a midpoint of hypotenuse AB. Then $\angle NOK$ is a right angle. Really, midline NO of $\triangle ABC$ is parallel to BC and $OK \perp BC$ (arcs CK and KB are equal).

Note that right triangles $\triangle ECO$ and $\triangle EAO$ are congruent (by side and hypotenuse). So EO is a bisector of $\angle AEC$.

Further, $\triangle AEEC$ is isosceles ($AE = EC$ as tangents to the circle). Then median EN is also a bisector. Therefore, EN and EO are both bisectors of the same $\angle AEC$; so E, N, O are colinear.

Furthermore, A, E, C and O belong to the same circumference ($\angle ECO = \angle EAO = 90°$). By power of the point we have

$$AN \times NC = EN \times NO,$$
$$AN \times NC = MN \times NK$$

which imply that

$$MN \times NK = EN \times NO,$$

meaning that M, K, E and O belong to the same circumference (by power of the point).

Then $\angle EMK = \angle EOK$ (subtended by the same arc). However, $\angle EOK = 90°$; therefore $\angle EMK = 90°$.

5 Answer: Mary has a winning strategy.

Consider John's number modulo 6.

Mary calls 2. If John continues to play, then his number was odd: $J \equiv 1, 3, 5 \mod 6$. His new number $J_1 = J - 2 \equiv 1, 3, 5 \mod 6$ is also odd.

Mary calls 3. So, if $J_1 \equiv 3 \mod 6$, Mary wins on her second move. So, after two moves John's number is $J_2 = J_1 - 3 \equiv 2, 4 \mod 6$ or $J_2 \equiv 2, 4, 8, 10 \mod 12$.

Mary calls 4. John continues to play, if $J_2 \equiv 2, 10 \mod 12$ or $J_3 = J_2 - 4 \equiv 10, 6 \mod 12$.

Mary calls 6. If $J_3 \equiv 6 \mod 12$ then $J_4 \equiv 0 \mod 12$, meaning that Mary wins. So, $J_4 \equiv 4 \mod 12$.

Mary calls 16. $J_5 \equiv 0 \mod 12$ and Mary wins. Note, that John's last number is not negative, for the most he subtracted is $2+3+4+6+16 = 31$.

There are other sequences of numbers of Mary's moves.

6 Answer: 2^{12}.

Let A be a 4×4-table consisting of "+" and "–".

Since it is allowed to change a sign in any cell (altogether with signs of all adjacent cells), we have 16 elementary transformations T_{ij} $(i, j = 1, \ldots, 4)$; all other transformations are compositions of elementary ones.

Note, that elementary transformations commute: if from table A we get table V applying some sequence of elementary transformations, then applying to A the same sequence, but in different order, we get V again. Also note, that changing sign in a cell (and in its neighboring cells) of table A twice we will get A again; therefore every elementary transformation needs to be applied no more than once.

Let T be a 4×4-matrix of transformation consisting of "0" and "1". The number 0(1) in cell (i, j) shows that elementary transformation T_{ij} is applied 0(1) times.

It is clear, that if table A and matrix T are given, then the resulting table V is uniquely defined. Note, that if we apply two transformations with matrices T and S, then resulting transformation corresponds to matrix $T + S$ (corresponding elements are added modulo 2).

(i) First, let us get an upper estimate. One can check that the following matrices do not change a table:

$$\begin{bmatrix}0110\\1001\\1001\\0110\end{bmatrix}, \begin{bmatrix}1111\\0110\\0110\\1111\end{bmatrix}, \begin{bmatrix}1001\\1111\\1111\\1001\end{bmatrix}, \begin{bmatrix}0000\\0000\\0000\\0000\end{bmatrix}, H = \begin{bmatrix}0001\\0011\\0101\\1110\end{bmatrix}, G = \begin{bmatrix}0010\\0111\\1000\\1011\end{bmatrix};$$

from matrix H we can get 3 more matrices with the same property by 90° rotations; from matrix G we can get 7 more matrices with the same property by rotations and a mirror reflection. Altogether, we have at least 16 (2^4) matrices P_α ($\alpha = 1, \ldots, 16$), which preserve tables.

Now let us divide all transformation matrices into equivalence classes in the following way: $T \sim S$ if applied to table A both produce the same result. Note that for any matrix of transformation S and any $\alpha = 1, \ldots, 16$ we have $S \sim S + P_\alpha$. So each equivalence class contains at least 2^4 elements and since there are 2^{16} matrices of transformations, there are at most $2^{16}/2^4 = 2^{12}$ different equivalence classes. This means that table A can generate no more than 2^{12} different tables.

(ii) Let us get a lower estimate. Let us color our table as a chess board with white top-left corner.

1. Note that any table could be transformed into a table with "$-$" in all black cells (if some black cell contains "+" we can change it to "$-$" without affecting all other black cells).

2. Now we show how with some special transformations we can make "$-$" in 4 white cells of the lower half of our table without affecting black cells. Let us consider the following matrices of transformations:

$$S_1 = \begin{bmatrix}0100\\1110\\0100\\0000\end{bmatrix}, \quad S_2 = \begin{bmatrix}1100\\1000\\0000\\0000\end{bmatrix}, \quad S_3 = \begin{bmatrix}0100\\1110\\0101\\0011\end{bmatrix}, \quad S_3 = \begin{bmatrix}1100\\1010\\0111\\0010\end{bmatrix}.$$

One can check that applying transformations with matrices S_1, S_2, S_3, S_4 we change signs only in cells, marked by 1 (all of them are white):

$$I_1 = \begin{bmatrix} 0000 \\ 0101 \\ 0000 \\ 0100 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1010 \\ 0000 \\ 1000 \\ 0000 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 0000 \\ 0100 \\ 0000 \\ 0001 \end{bmatrix}, \quad I_3 = \begin{bmatrix} 1000 \\ 0000 \\ 0010 \\ 0000 \end{bmatrix}.$$

Note that each of matrices I_j has exactly one "1" in the lower half-table. Therefore, if table A has "+" in some white cells of lower half-table, we can change them into "−" applying corresponding transformations S_i without affecting black cells.

Now, we have "−" in all cells except 4 white cells in the upper half-table. Thus we can transform A into one of 16 tables of this type; call them canonical tables.

Inversely, if one can reduce table A to canonical table V, one can restore A from V by the same transformation. We already proved in (i) that each table can be transformed into no more than 2^{12} tables; since there is 2^{16} tables and only 2^4 canonical tables, each canonical table can be transformed into exactly 2^{12} tables.

Therefore, *every table can be transformed into exactly* 2^{12} *tables.*

7 Answer: No.

Solution. Let us introduce *degree of vertex* P, the number of segments issued from P.

Let us assume that degrees of all vertices are even.

Lemma. *Let degrees of all vertices be even. Then one could paint all the triangles into two colors so that every two triangles with a common side would have different colors.*

Proof. Let us consequently paint adjacent triangles into opposite colors, every time connecting the centers of consequent triangles by a curve passing through their common side.

Assume that on some step we painted a triangle and found that an adjacent triangle had been already painted into the same color. Connecting centers of conflicting triangles we get a closed path, intersecting an odd number of segments; each of them is intersected only once. This path bounds some region D.

Consider directed segments issued from vertices belonging to D. Their total number i equals the sum of degrees of vertices belonging to D and is even by assumption. On the other hand, the number of directed segments with both ends in D is also even because each such directed segment is paired with the opposite one. Therefore the total number of (directed) segments intersecting our path must be also even.

This contraction proves lemma. □

Let us paint triangles according to Lemma. Due to the assumption that vertices of the square have even degrees as well, all the "boundary" triangles are painted in the same color, say, white.

Let W and B be the numbers of white and black triangles respectively. We assume that every inner segment has two sides; one is colored in black and the other in white colors. Then the total number of white sides is $3W$ while the total number of black sides is $3B$. Note that xactly 4 white sides do not have black counterparts; they are sides of the square. So, $3(W - B) = 4$ which is impossible. Contradiction.