Juniors
(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper **Spring 2003.**

1 ANSWER: Yes.

EXAMPLE. Consider quadratic equation $x^2 + 5x + 6 = 0$. It could be transformed into one of the following four equations:

(a) $x^2 + 5x + 6 = 0$ (roots $-2, -3$;

(b) $x^2 + 5x - 6 = 0$ (roots $-6, 1$);

(c) $x^2 - 5x + 6 = 0$ (roots $2, 3$);

(d) $x^2 - 5x - 6 = 0$ (roots $6, -1$).

2 The longest side of the triangle is a chord of a circumscribed circle and thus it does not exceeds its diameter: $a \leq 2R$. Projection of incircle onto the shortest altitude is contained strictly inside of the projection of the triangle onto this altitude. So $2r < h$. Since all the numbers are positive we can multiply these inequalities: $2r \cdot a < h \cdot 2R$ which implies $a/h < R/r$.

3

(a) ANSWER: Yes. Let us assign to i-th team a number $a_i = 0$, if prior to the game it already played even numbers of games and $a_i = 1$ otherwise. Note, that a_i changes after each game in which i-th team participated.

Assume, that all games were "even", meaning that prior to the game both teams had the same parity.

Consider the sum $A = a_1 + a_2 + \cdots + a_{15}$ of the parities of all teams. After each game played by two teams with the same parity A changes by $\pm 2 \equiv 2 \mod 4$.

Initially we had $a_1 = a_2 = \cdots = a_{15} = 0$, therefore $A = 0$. In the end we have $a_1 = a_2 = \cdots = a_{15} = 0$ (each team played an even number of games (14)) and again $A = 0$.

Since the total number of games $15 \cdot 14/2 = 105$ is odd, so in the end of the tournament $A \equiv 2 \mod 4$.

Contradiction.

(b) ANSWER: Yes. We will construct an example of a tournament with one "odd" game. Let us consider a graph, in which vertices represent teams and edges represent games. It is enough to draw edges in such a way that every time (but one) we connect the vertices of the same parity. Let us split all the vertices into three sets of five: $A_1, A_2, \dots A_5$; $B_1, \dots, B_5$; $C_1, \dots, C_5$. We proceed in three steps:

(i) *Step 1.* Let us connect all vertices in each set in the following order: $1-2$, $3-4$, $2-3$, $2-5$, $1-5$, $1-3$, $1-4$, $4-5$, $2-4$, $3-5$. One can check that each time we connect vertices of the same parity and in the end of this step all vertices have parity 0.

(ii) *Step 2.* Now, consider a cycle $A_1B_1C_1A_2B_2C_2 \dots A_5B_5C_5$. Let us connect vertices in order $A_1 - B_1$, $C_1 - A_2$, $\dots$, $A_5 - B_5$ (the same parity 0), $B_5 - C_5$ (opposite parities - *the only odd connection*), then $C_5 - A_1$, $B_1 - C_1$, $\dots$, $C_4 - A_5$ (the same parity 1). Note, that now all the vertices have parity 0.

(iii) *Step 3.* Now consider 5 sequences of five connections:

$$A_1 - B_1, A_2 - B_2, \dots, A_5 - B_5;$$
$$A_1 - B_2, A_2 - B_3, \dots, A_5 - B_1;$$
$$A_1 - B_3, A_2 - B_4, \dots, A_5 - B_2;$$
$$A_1 - B_4, A_2 - B_5, \dots, A_5 - B_3;$$
$$A_1 - B_5, A2 - B_1, \dots, A_5 - B_4.$$

We already made the first sequence. With each sequence the parities of vertices $A_1, \dots, B_5$ change; so after 4 sequences executed parities are restored to 0. Now all connections $A_i - B_j$ are done.

In the same way we make remaining 20 connections of $B_i - C_j$ and then remaining 20 connections $C_i - A_j$.

(b)$'$ *Second solution.* We construct an example for each $n = 4k - 1$, applying induction by k. For $k = 1, n = 3$ we make connections $1-2$, $2-3$, $3-1$ with only second connection odd.

Let us assume that the statement has been proven for $n = 4k - 1$; we will prove it for $n = 4k+3$, proceeding from k to $k+1$. So, we add extra 4 points. Already we have n old points connected between themselves with one odd connection. Now all these vertices are even because each of them is connected with $n - 1 = 4k - 2$ others. Let us split old points in $k - 1$ quartets and one triplet. Consider an old quartet $Q_1, \dots, Q_4$ and a new one $N_1, \dots, N_4$ and make the following 4 sequences of 4 connections each:

$$Q_1 - N_1, Q_2 - N_2, Q_3 - N_3, Q_4 - N_4;$$
$$Q_1 - N_2, Q_2 - N_3, Q_3 - N_4, Q_4 - N_1;$$
$$Q_1 - N_3, Q_2 - N_4, Q_3 - N_1, Q_4 - N_2;$$
$$Q_1 - N_4, Q_2 - N_1, Q_3 - N_2, Q_4 - N_3.$$

After each sequence the parities of all points in both quartets change and in the end they are restored. Let us repeat this procedure, connecting points $N_1, \dots, N_4$ with all old points except T_1, T_2, T_3 (last triplet).

Then we make connections $T_1 - N_1$, $T_2 - N_2$, $T_3 - N_3$ (all points but N_4 become odd). Now connect:

$$N_2 - N_3, N_3 - N_4, N_4 - T_3, N_2 - N_4, N_4 - T_2, N_1 - N_2, N_1 - N_4, N_4 - T_1, N_1 - N_3.$$

One can check easily that all these connections are even. Each new points is connected with other points.

4 Answer: if n is prime, Second Player has a winning strategy; otherwise First Player has.

(i) Let n be a prime number. Let First Player eat a triangle with side k. Leftover is a trapezoid with sides $(k, n-k, n, n-k)$. Denote $a = \max(k, n-k)$ $b = \min(k, n-k)$. Note that $a \neq b$ because $\gcd(a, b) = \gcd(n, n-k) = 1$. Second Player eats a triangle with the side $n-k$, leaving the parallelogram with sides a and b.

(A) Now, if First Player eats triangle with side less than b, then Second Player repeats his move symmetrically (with respect to the center of the parallelogram), and wins since First Player has no move.

(B) If First Player eats triangle with side b, leftover is the trapezoid with sides $(a-b, b, a, b)$, where $\gcd(a-b, b) = \gcd(a, b) = 1$. The game is over when $a = b = 1$, meaning that the last triangular chip is left after First Player's move. Therefore, Second Player wins.

(ii) Let n be a composite number, p any prime divisor of n, $n = kp$. First Player eats triangle with side p. Consider two cases:

(A) If Second Player eats triangle with a side, not equal to $n-p$, then First Player eats triangle with side 1 and wins.

(B) If Second Player eats a triangle with the side $n-p$, then leftover is a parallelogram with sides p and $(k-1)p$. First Player eats the triangle with side p. Again, if Second Player eats triangle with side, not equal to p, then First Player eats triangle with side 1 and wins. So, in the end, after First Player's move, leftover is a triangle with side p. We are in the situation (A) now; however, Second Player has the first move, therefore, he loses.

5 Answer:21

(I) *Example: Fig. 1*

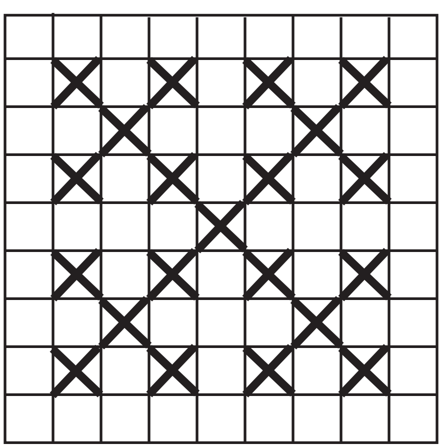


Fig. 1

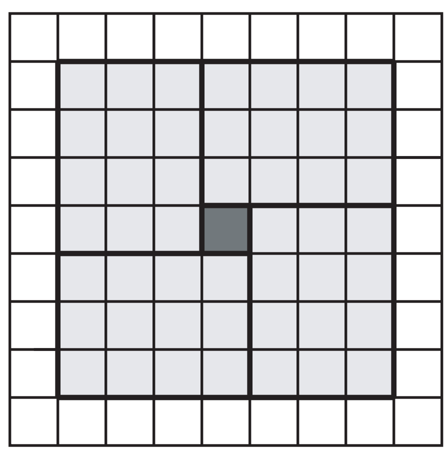


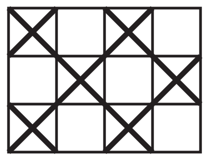
Fig. 2

(II) *Let us prove, that 21 is a maximum.*

First, note that cutting any square on the border results in the board falling apart. Cutting any two adjacent squares also results in failure. Let us divide 7×7 board without central square into four rectangles 3×4 as on Fig. 2. Let us show, that no more then five squares can be cut in each rectangle. Assume, that it is possible to cut at least six. Since row of 3×4 rectangle contains no more than two squares cut, so we have exactly two squares cut in each row. Consider two cases:

(i) The first line is cut (X- -X). Then in the second row only one square could be cut.

(ii) The first row is cut (X-X-) or (-X-X) then second line is cut (-X-X) or (X-X-) and the third line is cut like (X-X-) or (-X-X) again:



However, this results in the board falling apart. Contradiction.

(II)′ *Second proof that 21 is a maximum.* First of all, we cut all 81 squares. Let us prove that one needs to repair at least 60 squares in order to restore integrity of the board. Really, all squares are cut, the board splits into 180 pieces (9 triangles along each border and one diamond at each pair of adjacent squares; there are 8 pairs of adjacent squares in each row and column of the board; so we get $\big(4 \times 9 + (9+9) \times 8 = 180\big)$ of pieces.

Repairing one square we join no more than 4 different pieces, decreasing their total number by no more than 3. So, to get 1 piece we need to repair at least $\left\lceil \frac{179}{3} \right\rceil = 60$ squares.

6 Let O be the center of incircle, K and L tangency points with sides AD and BC respectively.

Solution 1. We start from two following statements:

Lemma 1. *Points K, E, O and L are colinear.*

Proof (see Fig.1 on next page). Note that OK and OL are perpendicular to bases of the trapezoid and thus are parallel. So, O belongs to KL. One can assume with no loss of the generality that $AD > BC$ (if $AD = BC$ our trapezoid is a rhombus and $\angle AED = 90°$).

Let N be a point of intersection of AB and CD. Let K' be a point of tangency of incircle of $\triangle BCN$ with side BC. From the property of tangents (drawn from the same point to the circle) we have

$$BK + BN = CK + CN$$
$$K'C + BN = p$$

where p is a half-perimeter of $\triangle BCN$. So, $BK = CK'$.

Note that $\triangle BEC \simeq \triangle DEA$ ($BC \parallel AD$). Then $\dfrac{BE}{ED} = \dfrac{BC}{AD}$ and therefore $\dfrac{BE}{ED} = \dfrac{BK}{LD}$. This implies that $\triangle BKE \simeq \triangle DLE$ ($\angle KBE = \angle LDE$ and $\dfrac{BE}{ED} = \dfrac{BK}{LD}$). Then $\angle BEK = \angle DEL$ which means that points K, E, L are colinear. $\square$

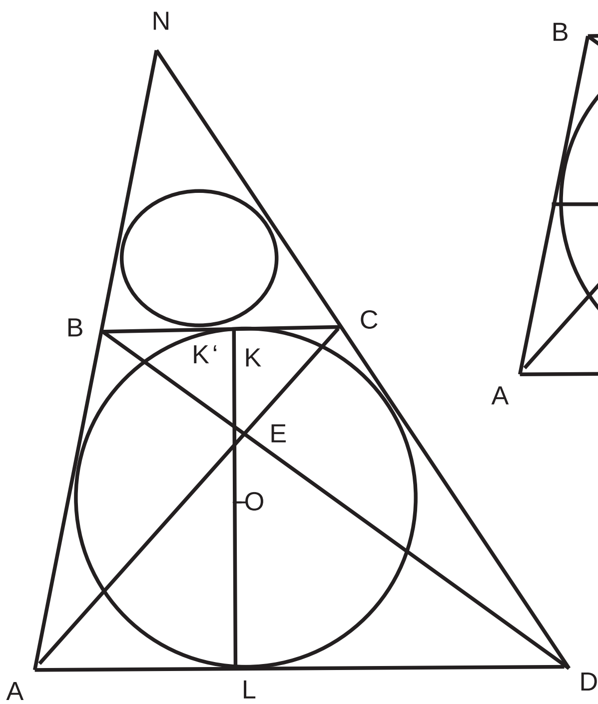


Fig. 1

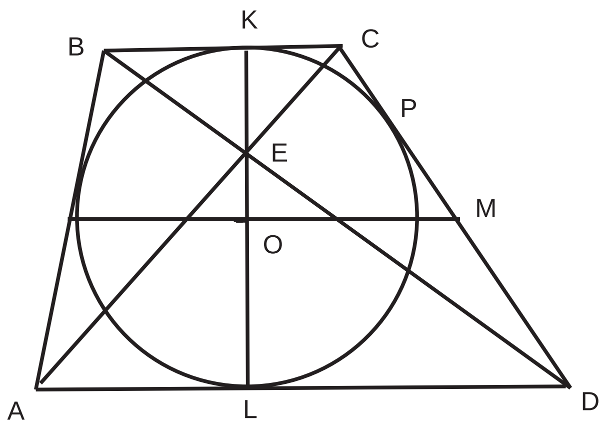


Fig. 2

Lemma 2. *Let S be a midpoint of side PQ of $\triangle PQR$. If $RS = \frac{1}{2}PQ$ then $\angle PRS = 90°$. If $RS < \frac{1}{2}PQ$ then $\angle PRS > 90°$.*

Proof. Consider a circumference with diameter PQ. Then R belongs to this circumference in the former case and lies inside of it in the latter case. $\square$

Let M be a midpoint of CD (see Fig. 2); then OM is a midline of trapezoid $KCDL$, and therefore it is parallel to its bases and is equal to $(KC + LD)/2 = (PC + PD)/2 = CD/2$ where P is a point of tangency with CD. Then by lemma 2 $\angle COD = 90°$ and O belongs to a circumference with diameter CD and a center M. Since $MO \parallel CK$, therefore $MO \perp KL$, we conclude that KL is tangent to this circle at O. Then all points of KL (but O) are outside of this circumference. Therefore $\angle DEC$ does not exceed $90°$, so $AED \geq 90°$.

Solution 2. Extending AD beyond A (see Fig. 3), we choose point D' such that $AD' = BC$. Also extending BC beyond B we choose point C' such that $BC' = AD$. Then $CC'D'D$ is a parallelogram. Select point N on CC', such that $C'N = D'A = BC$.

Since $AC'BD$ is a parallelogram ($C'B = AD$, $C'B \parallel AD$) then $C'A \parallel BD$. Therefore $\angle BEC = \angle C'AC$. So we need to prove that $\angle C'AC \geq 90^\circ$. Let M be a midpoint of CC'; then M is a midpoint of NB. Then $CC' = AD + BC = AB + CD$ (property of circumscribed quadrilateral), and $CD = AN$ (because $ANC'D'$ is a parallelogram) and $AB + AN \geq 2AM$ (triangle inequality). Then $\angle C'AC \geq 90^\circ$ due to lemma 2.

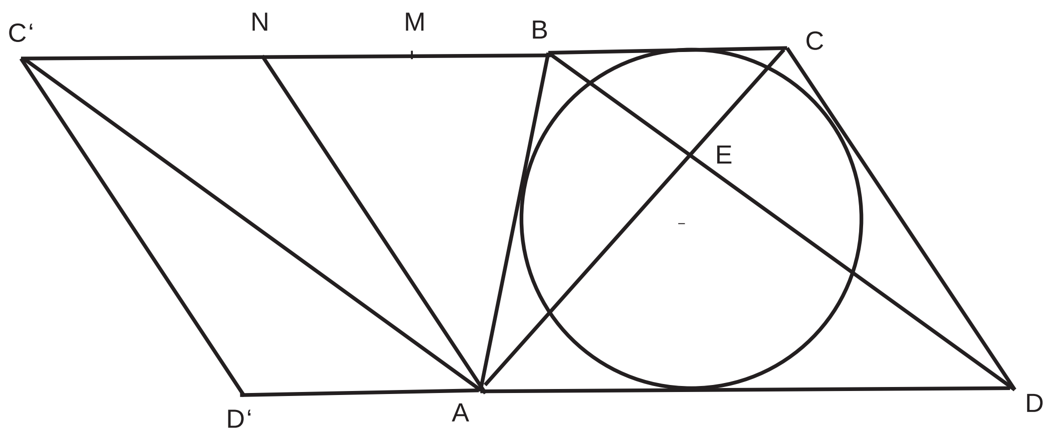


Fig. 3