International Mathematics
TOURNAMENT OF THE TOWNS

Senior O-Level Paper **Fall 2003.**
Solutions

1. Let us denote the greatest odd divisor of number K as god(K). Obviously, god$(K) \leq K$. Assume, that god$(l) =$ god(m), $l, m \in [n+1; 2n]$, $l < m$. Then $m \geq 2l$, which is impossible.

   Therefore, for all (n) numbers from $[n+1, 2n]$ their corresponding greatest odd divisors are distinct. It means, that the set $\{\text{god}(m), n+1 \leq m \leq 2n\}$ coincides with $\{1, 3, \ldots, 2n-1\}$. Then, the sum in question is $1 + 3 + \cdots + 2n + 1 = n^2$.

2. Let us solve the problem for a $(2n+1) \times (2n+1)$-square. Note that all unit boundary squares $(8n)$ should be drawn (otherwise there would be "holes" in a frame). Now we have a $(2n-1) \times (2n-1)$ square with the frame. Let it be colored as a chess board, with black squares at the angles. Drawing only white squares would give us the whole picture in question. The number of white squares is equal to $\big((2n-1)^2 - 1\big)/2$. Then the total number of the unit squares used equals $8n + ((2n-1)^2 - 1)/2 = 360$, if $n = 25$.

   Let us show that it is indeed the minimal number. Let us tile a $(2n-1) \times (2n-1)$ square with dominos (2×1 rectangles). Then we use $\big((2n-1)^2 - 1\big)/2$ dominos with 1 unit square left. In each domino we have to draw at least one square, so the least number of squares drawn is $\big((2n-1)^2 - 1\big)/2$.

3. Let Customer give all his money to Salesman. The value of change could vary from 0 (if Cat is a "gift") to 1999 rubles and we show that any integer value could be created within a set of bills. It is enough to solve the problem in the case of the following (minimal) set of sixteen bills $\{1000, 500, 100, 100, 100, 100, 50, 10, 10, 10, 10, 5, 1, 1, 1, 1\}$. Actually, representing 1999 as 1000+900+90+9 one can see that in the "minimal" case we must have exactly one 1000 ruble bill, one 500 ruble bill, four 100 ruble bills, one 50 ruble bill, four 10 ruble bills, one 5 ruble bill and four 1 ruble bills. Now, notice that if a transaction could be made with the minimal set of bills, then it could be also made with any other set of bills. Actually, each smaller nomination divides every larger one. Therefore, if in an arbitrary set we have more than, say, four ruble bills, we wrap five rubles by a rubber band and consider it as a 5 ruble bill and so on. So, any set of bills could be reduced to the minimal set.

   It is easy to see that any value of change in the form $ABCD$, where $A = 0, 1$ and B, C, D are any digits from 0 to 9, could be paid with the minimal set of bills.

4. Lemma. The area of a quadrilateral, placed into a circle with radius R does not exceed $2R^2$.

   Proof. Let $ABCD$ be the quadrilateral in mention, and O the center of the circle. Then, Area$(\triangle ABO) = \frac{1}{2} AO \cdot BO \cdot \sin(\angle AOB) \leq \frac{1}{2} R^2$ and

$$\text{Area}(ABCD) = \text{Area}(\triangle ABO) + \text{Area}(\triangle BCO) + \text{Area}(\triangle CDO) + \text{Area}(\triangle DAO) \leq 2R^2.$$

   Let O be the center of the given square $KLMN$.

(a) Notice, that points A, B, C, D lie on semicircles with diameter 1. Let S be the midpoint of side KL of $\triangle AKL$. Then $AO \leq OS + AS = \frac{1}{2} + \frac{1}{2} = 1$. In a similar way, we have $BO \leq 1$, $CO \leq 1$, $DO \leq 1$, meaning that points A, B, C, D lie in a circle with the center O and radius 1. Then according to Lemma, Area$(ABCD) \leq 2$.

(b) Let O_1 be the center of an incircle of $\triangle AKL$. Then, for any position of A $\angle KO_1L = 135°$ (O_1 is the intersection of bisectors of $\triangle KAL$); therefore O_1 lies on the circumference with center O and radius $\sqrt{2}/2$. Then $OO_1 = \sqrt{2}$. In similar way we have $OO_2 = OO_3 = OO_4 = \sqrt{2}$. Therefore, according to the Lemma, the Area$(O_1O_2O_3O_4) \leq 1$.

5. ANSWER: yes. Under "development" we understand that the figure could be flattened on a plane (folded, but not teared or crumpled).

EXAMPLE. Consider a Cartesian coordinate system and an "envelop" $PABQD$ (consisting of two layers with PQ open; the first layer is a rectangle $PABQ$ and the second layer consists of triangles PAD, ADB and BDQ), where $D(0,0), A(-1/2, \sqrt{3}/2), B(1/2; \sqrt{3}/2), Q(1/2; 0), P(-1/2, 0), C(0,0)$.

Now, let us consider this figure in a space. Pull point D up while pushing point C down until points P and Q coincide (in the mid point of segment CD). We get tetrahedron $ABCD$, with edge CD cut. Reversing the process, we could develop a (regular) tetrahedron $ABCD$ into a planar figure. It is easy to see that

(a) this procedure could be implemented in a space.

(b) a tetrahedron is a closed 3-D figure, so by cutting only one edge it is impossible to develop it in one layer.