Seniors
(Grades 11,12)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2003.**

1 [**4**] Smallville is populated by unmarried men and women, some of them are acquainted. Two city's matchmakers are aware of all acquaintances. Once, one of matchmakers claimed: "I could arrange that every brunette man would marry a woman he was acquainted with". The other matchmaker claimed "I could arrange that every blonde woman would marry a man she was acquainted with". An amateur mathematician overheard their conversation and said "Then both arrangements could be done at the same time!" Is he right?

2 [**4**] Prove that every positive integer can be represented in the form

$$3^{u_1} \cdot 2^{v_1} + 3^{u_2} \cdot 2^{v_2} + \ldots + 3^{u_k} \cdot 2^{v_k}$$

with integers $u_1, u_2, \ldots, u_k, v_1, \ldots, v_k$ such that $u_1 > u_2 > \ldots > u_k \geq 0$ and $0 \leq v_1 < v_2 < \ldots < v_k$.

3 [**6**] An ant crawls on the outer surface of the box in a shape of rectangular parallelepiped. From ant's point of view, the distance between two points on a surface is defined by the length of the shortest path ant need to crawl to reach one point from the other. Is it true that if ant is at vertex then from ant's point of view the opposite vertex be the most distant point on the surface?

4 [**7**] In a triangle ABC let H be the point of intersection of altitudes , I the center of incircle, O the center of encircle, K the point where incircle touches BC. Given, that IO is parallel to BC, prove that AO is parallel to HK.

5 [**7**] Two players in turns play a game. Each player has 1000 cards with numbers written on them; namely, First Player has cards with numbers $2, 4, \ldots, 2000$ while Second Player has cards with numbers $1, 3, \ldots, 2001$. In each his turn, a player chooses one of his cards and puts it on a table; the opponent sees it and puts his card next to the first one. Player, who put the card with a larger number, scores 1 point. Then both cards are discarded. First Player starts. After 1000 turns the game is over; First Player has used all his cards and Second Player used all but one. What are the maximal scores, that players could guarantee for themselves, no matter how the opponent would play?

6 [**7**] Let O be the center of insphere of a tetrahedron $ABCD$. The sum of areas of faces ABC and ABD equals the sum of areas of faces CDA and CDB. Prove that O and midpoints of BC, AD, AC and BD belong to the same plane.

7 A $m \times n$ table is filled with signs "+" and "−". A table is called irreducible if one cannot reduce it to the table filled with "+", applying the following operations (as many times as one wishes).

a) [**3**] It is allowed to flip all the signs in a row or in a column. Prove that an irreducible table contains an irreducible 2×2 sub table.

b) [**6**] It is allowed to flip all the signs in a row or in a column or on a diagonal (corner cells are diagonals of length 1). Prove that an irreducible table contains an irreducible 4×4 sub table.