International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2003.**
Solutions

1. ANSWER: yes.

   EXAMPLE: Placing 5 into each square of two 4×3-faces, 8 into each square of two 5×3-faces and 9 into each square of two 4×5-faces , we satisfy all the given requirements.

2. Assume that a 7-gon is convex. In order for a polygon to be regular, one needs to prove equalities of all its sides and angles. Let us color the sides of the polygon with white, the first set of equal diagonals with blue, and the second set of equal diagonals with red. All triangles created by two blue and one red sides are equal (S-S-S). Therefore the next two sets of angles are equal:

   (a) Angles at the bases of isosceles triangles mentionned (let us denote them by α).
   (b) Angles at the vertices of isosceles triangles mentionned (let us denote them by β).

   Consider the triangles created by three different colors. All the angles opposed to white sides are equal (each equals $\alpha-\beta$). Then all sides of the polygon are equal (S-A-S).

   Now, all the triangles, created by two white sides and one blue side are equal (S-S-S). Thus all angles of the polygon are equal.

   Therefore, the polygon is regular.

3. Let us denote the greatest odd divisor of number K as god(K). Obviously, god$(K)\le K$. Assume, that god$(l)=$ god(m), $l,m\in[n+1;2n]$, $l<m$. Then $m\ge 2l$, which is impossible.

   Therefore, for all (n) numbers from $[n+1,2n]$ their corresponding greatest odd divisors are distinct. It means, that the set $\{\text{god}(m), n+1\le m\le 2n\}$ coincides with $\{1,3,\dots,2n-1\}$. Then, the sum in question is $1+3+\dots+2n+1=n^2$.

4. Let us note that

   (a) from each point emanates exactly $(n-1)$ segments (each pair of points is connected).
   (b) from each point emanates either 2 blue/red segments or 2 blue and 2 red segments (each broken line is closed, without intersection and there are no isolated points).

   Therefore, we have 2 possible cases to consider:

   (i) n=3.
   (ii) n=5.

   Case (i) is trivial, corresponding to a triangle with all sides of the same color.

   Case (ii) is also possible:

   EXAMPLE. Consider points $A(0;4); B(-4;-4); C(4;-4); D(-1;0); E(1;0)$. Connect them by red segments in the order A,B,C,D,E,A and by blue segments in the order A,D,B,D,C,A.

5. ANSWER: 50.

Let us prove first that the arrangement in question is impossible, if $N < 50$. Case $N = 49$ means that checker "25" stays on its initial place, so the only possible move is that checker "24" jumps through checker "25" on the 26-th place (its final position). After that any movement to the right is impossible.

Example, that $N = 50$ works.

(a) Checker "25" moves on the 26-th place (its final position).
(b) Checker "23" jumps through "24" and "25", then moves on the 28-th place (its final position).
(c) Checker "21" jumps through "22", "24", "25" and "23", then moves on the 30-th place (its final position)

and so on...

Finally, checker "1" jumps through "2", "4", ... "22", "24", "25", "23",..., "3", then moves on the 50-th place (its final position).

Now,

(a) Checker "2" moves one space to the right and then jumps through "4", ..., "22", "24", "25", "23",...,"3" occupying the 49-th place (its final position).
(b) Checker "4" moves one space to the right and then jumps through "6", ..., "22", "24", "25", "23",...,"5" occupying the 47-th place (its final position)

and so on ...