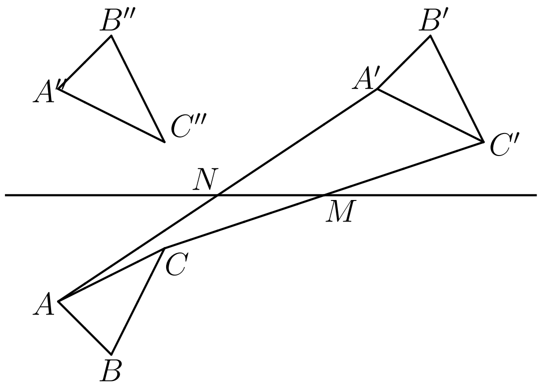
Solution to Senior O-Level Spring 2002

1. If either x or y is odd, x^2+xy+y^2 is also odd. Hence they are both even. If one is a multiple of 10 and the other is not, x^2+xy+y^2 is not a multiple of 10. Suppose both x and y are not multiples of 10. Then x^2 and y^2 end in 4 or 6, while xy cannot end in 0. So we cannot have one ending in 4 and the other in 6. If x^2 and y^2 both end in 4 or both end in 6, then xy must also end in 4 or 6. It follows that the only possibility is for both x and y to be multiples of 10, so that x^2+xy+y^2 will indeed be a multiple of 100.

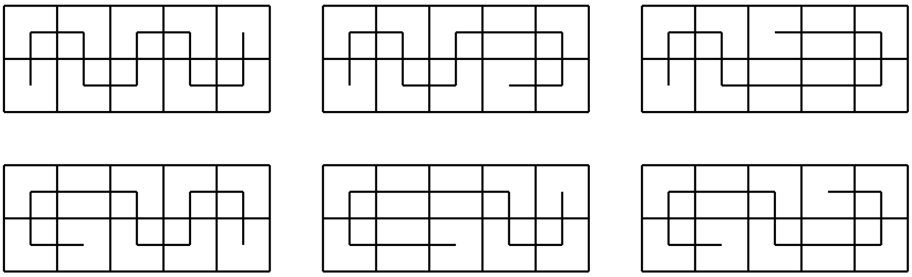
2. Let M be the midpoint of AA' and N be the midpoint of CC'. Then A and A' are equidistant from MN, as are C and C'. Let $A''B''C''$ be the reflection of ABC across MN. Then A and A'' are equidistant from MN, as are C and C''. Hence $A'A''$ and $C'C''$ are both parallel to MN. Now $A''B''C''$ is congruent to ABC and opposite in orientation. Hence it is congruent to $A'B'C'$ and in the same orientation. It follows that $A'B'C'$ and $A''B''C''$ may be obtained from each other by a translation in the direction parallel to MN. Hence B' and B'' are equidistant from MN. It follows that so are B and B', so that the midpoint of BB' indeed lies on MN.



3. The only possible groupings are (126,345), (136,245), (146,235), (156,234) and (236,145). First weigh 146 against 235. If they balance, the task is accomplished. If 146 is heavier, then 156 will be heavier than 234. Then we weigh 136 against 245. If they balance, the task is accomplished. If 136 is heavier, then 236 will be heavier than 145. Hence 126 must balance 345. If in the first weighing 146 is lighter, then 136 will be lighter than 245, 126 will be lighter than 345 and 145 will be lighter than 236. Hence 156 must balance 234.

4. We first solve the problem for a 2×5 table. Each successful placement of the numbers is replaced with a continuous path from one number to the next. Suppose first that 1 and 10 are also adjacent, so that the path could have linked up to form a cycle. The cycle could be broken up in any of 10 places. Hence there are 10 paths of this kind. Suppose now that 1 and 10 are not adjacent, so that we have an open path. We classify them according to whether the vertical segments are in one, two or three groups, where vertical segments on adjacent columns are considered to be in the same group. Note that apart from a path obtained from the cycle, each end column must contain a vertical segment.

For paths in which all the vertical segments are in one group, this means that each column must contain a vertical segment. This path, shown in the first figure below, is unique if we assume for now that the left endpoint must be on the bottom row. For paths in which the vertical segments are in two groups, we cannot have each groups containing at least two segments. On the other hand, if each contains exactly one segment, then we have a path obtainable from the cycle. Hence exactly one group contains exactly one segment. Continuing to assume that the left endpoint is on the bottom row, we have four paths as shown in the next four figures. Finally, for paths in which the vertical segments are in three groups, each end group must contain exactly one segment. This unique path is shown in the last figure. Lifting the restriction that the left endpoint be on the bottom row, we have 12 paths. Along with the 10 obtained from the cycle, we have a total of 22. Since each path may be traversed in either direction, there are 44 different placements of the numbers.



We now solve the given problem. There are 100 paths obtainable frmo the cycle. Among the others, there are 2 in which all vertical segments are in one group. For those in two groups, the larger group may consist of 2 to 48 segments. Since the larger group may be on either end, and the left endpoint may be on either row, there are $4 \times 47 = 168$ paths of this type. Finally, for those in three groups, the middle group may consist of 1 to 46 segment. For $1 \leq k \leq 46$, these segments may have $46 - k$ possible locations. Since the left endpoint may be on either row, the total number of paths of this type is $2(46 + 45 + \cdots + 1) = 2162$. Thus the total number of paths is $100 + 2 + 168 + 2162 = 2432$, and the total number of different placements of the number is $2 \times 2432 = 4864$.

5. The diagram below shows a regular triangular prism covered without overlap by three equilateral triangles of different sizes.