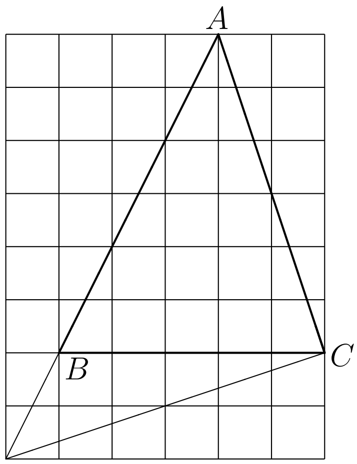
Solution to Senior A-Level Spring 2002

1. First, note that we have

$$\begin{aligned}\tan A+\tan B+\tan C &= \tan A+\tan B-\frac{\tan A+\tan B}{1-\tan A\tan B}\\ &= (\tan A+\tan B)\left(1-\frac{1}{1-\tan A\tan B}\right)\\ &= -\frac{\tan A+\tan B}{1-\tan A\tan B}\tan A\tan B\\ &= \tan A\tan B\tan C.\end{aligned}$$

Let $\tan A=a,\ \tan B=b$ and $\tan C=c$ where $a,\ b$ and c are integers such that $a+b+c=abc$. ABC cannot be a right triangle. Suppose $\angle A$ is obtuse. Then a is negative while b and c are positive. If $b=c=1$, then $abc=a<a+2=a+b+c$. Any increase in the values of b or c will increase that of $a+b+c$ while decrease that of abc. It follows that ABC is an acute triangle, so that $a,\ b$ and c are all positive. We may assume that $a\le b\le c$. Then $abc=a+b+c\le 3c$, so that $ab\le 3$. We cannot have $a=b=1$. Hence $a=1,\ b=2$ and $c=3$. Finally, the diagram below shows a triangle ABC with $\tan A=1,\ \tan B=2$ and $\tan C=3$.



2. Consider the points $A(a,a^3)$ and $B(b,b^3+b+1)$ where $a>b>0$. We wish to choose a and b such that $a-b<\frac{1}{100}$ while $a^3=b^3+b+1$. Let $t=a-b>0$. ¿From $(b+t)^3=b^3+b+1$, we have $3tb^2-(1-3t^2)b-(1-t^3)=0$. If $t<\frac{1}{100}$, the constant term of this quadratic equation is negative, so that it has one positive root and one negative root. Thus a and b can be chosen so that $AB<\frac{1}{100}$.

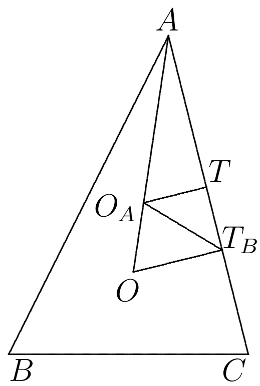
3. Let the sequence be $\{a_n\}$ and let S_n denote the sum of all the terms up to but not including a_n. For $n\ge 2002$, a_n is a divisor of S_n. Hence there exists a positive integer d_n such that $a_n=\frac{S_n}{d_n}$. Then $S_{n+1}=S_n+a_n=\frac{(d_n+1)S_n}{d_n}$. If $d_{n+1}\ge d_n+1$, then $a_{n+1}\le\frac{S_n}{d_n}=a_n$, and this contradicts the hypothesis that $\{a_n\}$ is strictly increasing. Hence $\{d_n\}$ is non-decreasing for $n\ge 2002$. However, this sequence cannot maintain a value $k>1$ indefinitely as otherwise $\{S_n\}$ becomes a geometric progression with common ratio $\frac{k+1}{k}$ starting from some term. However, k and $k+1$ are relatively prime, and we can only divide the first term of the geometric progression by k finitely many times. It follows that $d_n=1$ eventually.

4. We use induction on the number n of spectators. The case $n = 2$ holds as a single switch fixes the derangement. Suppose the result holds from 1 to n for some $n \geq 1$. Consider the next case with $n+1$ spectators. Let S_k be the spectators with the ticket k. Suppose S_{n+1} is in seat m for some $m \leq n$. If the spectators in seats m to $n+1$ constitute a derangement among themselves, we can appeal to the induction hypothesis. Otherwise, there exists a seat ℓ which is the first after seat m to be occupied by some S_x where $x \neq \ell - 1$. This means that for $m < k < \ell$, seat k is occupied by S_{k-1}. We perform a chain of switches from seat ℓ back to seat $m+1$, we still have a derangement since S_k is now in seat $k+2$ for $m < k < \ell$. This brings S_x to seat $m+1$ and we can now switch her with S_{n+1}, bringing the latter one seat closer to her correct place. We can now repeat the above process until S_{n+1} is in seat $n+1$, and then appeal to the induction hypothesis.

5. Since BCB_1C_1 is cyclic, triangles ABC and AB_1C_1 are similar. Tthe ratio of similarity is $\cos\alpha$ where $\alpha = \angle CAB$, since $AB_1 = AB\cos\alpha$. Let O be the incentre and r the inradius of ABC, and let T be the point of tangency of the incircle of AB_1C_1 with AC. Now $OT_B = r$, $O_AT = r\cos\alpha$, $AT = AT_B\cos\alpha$, $AT_B = r\cot\frac{\alpha}{2}$ and

$$TT_B = AT_B - AT = AT_B(1-\cos\alpha) = r\cot\frac{\alpha}{2}\left(2\sin^2\frac{\alpha}{2}\right) = r\sin\alpha.$$

Hence $O_AT_B = \sqrt{O_AT^2 + T_BT^2} = r$. By symmetry, the other sides of the hexagon are also equal to r.



6. If two adjacent cards are of the same suit, we say that there is a suit bond between them. If they are of the same rank instead, we say that there is a rank bond between them. By hypothesis, there is either a suit bond or a rank bond between two adjacent cards, and it cannot be both since each card is unique within a deck. So we have twelve columns each consisting of four horizontal bonds, and three rows each consisting of thirteen vertical bonds. We claim that in each row and each column, the bonds are of the same type. Assuming to the contrary that there are both suit bonds and rank bonds in a column. Then there is one of each kind in two adjacent rows. Of the four cards in question, let the top two be the Ace and King of Hearts. The bottom two are of the same rank. If this rank is Ace, then there is no bond between the King of Heart and the card below. Similarly, this rank cannot be King. Now not both cards at the bottom can be Hearts. Hence one of them will not have a bond with the card above. This justifies our claim. Considering the types of bonds for each of the three rows of vertical bonds, we have eight cases.

(i) All three rows are rank bonds. This yields the desired conclusion.

(ii) All three rows are suit bonds. This means that the 52 cards are in 13 groups of 4, with cards in the same group being of the same suit. This is impossible since 13 is not a multiple of 4.

(iii) Only the top and bottom rows are suit bonds. This means that we have 26 disjoint pairs of cards of the same suit. This is impossible since 13 is not a multiple of 2.

(iv) Only the top and bottom rows are rank bonds. Consider the 13 inside pairs of cards in the second and the third rows, with a suit bond between each pair. We may assume that the first pair are Spades. There must be a first pair which are not Spades, say Hearts. Consider first the case where the two outside cards in the first column are of the same suit, which cannot be Hearts. We may assume it is Clubs. Then the two outside cards on the column with Hearts inside must be Diamonds. When the inside pair change suits again, it must go from Hearts to either Spades or Clubs. It follows that each column of 4 cards have the same colour. However, there are 26 red cards and 26 is not a multiple of 4. Consider now the case where the two outside cards in the first column of different suits. Then they must be Diamonds and Clubs. Then the two outside cards on the column with Hearts inside must be Clubs and Diamonds. It follows that all the Spades and Hearts form 13 inside pairs, but there are 13 Spades and 13 is not a multiple of 2.

(v) Only the top two rows are suit bonds. We may assume that the top three cards in the first column are Spades and that the bottom card is Clubs. This remains the case until we encounter the first column of horizontal suit bonds. Then the four cards in the next column must all be red. It follows that the four cards in each column are of the same colour. However, there are 26 red cards and 16 is not a multiple of 4.

(vi) Only the bottom two rows are suit bonds. This is analogous to Case (v).

(vii) Only the top two rows are rank bonds. This means that there are 3 cards of the same rank in each column. Since there are only 4 cards of each rank, all 13 columns consist of different ranks. Hence the first row of horizontal bonds are are suit bonds, so that all columns of horizontal bonds are suit bonds. This forces all the vertical bonds in the bottom row to be rank bonds too, contrary to our assumption.

(viii) Only the bottom two rows rae rank bonds. This is analogous to Case (vii).

7. Let $a = \sqrt{6}$ and $b = sqrt3$. Suppose $\lfloor a^m \rfloor = \lfloor b^n \rfloor$ for some positive integers m and n. Denote their common value by k. Then $k^2 \leq 6^m < k^2 + 2k + 1$ and $k^2 \leq 3^n < k^2 + 2k + 1$. It follows that $2k \geq |6^m - 3^n| = 3^m |2^m - 3^{n-m}|$. Clearly, $n > m$ so that $|2^m - 3^{n-m}| \geq 1$. Hence $2k \geq 3^m$ so that $\frac{9^m}{4} \leq k^2 \leq 6^m$. Now the only positive integral values of m for which $\frac{1}{4} \leq (\frac{2}{3})^m$ holds are 1, 2 and 3. We have $\lfloor a \rfloor = 1$, $\lfloor a^2 \rfloor = 6$ and $\lfloor a^3 \rfloor = 14$. On the other hand, $\lfloor b \rfloor = 1$, $\lfloor b^2 \rfloor = 3$, $\lfloor a^3 \rfloor = 5$, $\lfloor a^4 \rfloor = 9$ and $\lfloor a^5 \rfloor = 15$. It follows that $\lfloor a^m \rfloor \neq \lfloor b^n \rfloor$ for any positive integers m and n.