Solution to Junior A-Level Spring 2002

1. We have

$$\begin{aligned} a^3+b^3+3abc-c^3 &= a^3+b^3+(-c)^3-3ab(-c) \\ &= (a+b+9-c))(a^2+b^2+(-c)^2-b(-c)-(-c)a-ab) \\ &= \frac{1}{2}(a+b-c)((b+c)^2+(c+a)^2+(a-b)^2). \end{aligned}$$

   This is positive since $a+b-c>0$ in a non-degenerate triangle.

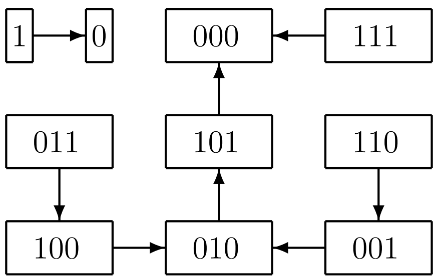
2. Initially, the four chips determine a rectangle, with chips of the same colour at opposite corners. After a move by the first player from such a position, there is no victory since the two white chips are in different rows and different columns. Moreover, the four chips will no longer determine a rectangle. However, the second player can restore this position in his move. Thus there is no victory for the first player.

3. Denote the area of the polygon P by $[P]$. Then

$$[BAD]=[ABEFD]-[BEFD]=[ABE]+[AEF]+[AFD]-3[BCD].$$

   In order to maximize $[BAD]$, BCD must have the smallest area among the four triangles whose area are four consecutive integers. The maximum value of $[BAD]$ is $[BCD]+1+[BCD]+2+[BCD]+3-3[BCD]=6$.

4. Denote by 0 a lamp which off and by 1 a lamp which is on. The following diagram shows that for $n=1$ or 3, there are no initial configurations which lead to perpetual light.



   For even n, the initial configuration 1001100110... will work since it will alternate with 0110011001.... For odd $n>3$, just add 010 to the previous configuration. It will alternate with 100 since the third light will not go on because of the fourth. Hence this part will alternate with 100, independent of the second part. In conclusion, perpetual light is possible for all n except 1 and 3.

5. Consider a convex n-gon. It is potentially $n-2$ triangles. Suppose it is cut into a convex a-gon and a convex b-gon. Then the number of potential triangles is $a-2+b-2$. There are essentially three way of cutting the convex n-gon: through two vertices, through one vertex or through no vertices. In the first case, we have $a+b=n+2$, so that the number of potential triangles is $n-2$ as before. No new obtuse angles can be created since no angle of the convex n-gon can be divided into two obtuse angles. In the second case, we have $a+b=n+3$, so that the number of potential triangles is $n-1$, an increase of 1. As before, the end of the cut through a vertex cannot create a new obtuse angle, but the other end which ends on a side can create one, but no more than one, new obtause angle. In the third case, we have $a+b=n+4$ so that the number of potential triangles is n, an increase of 2. Each end of the cut can create one, but no more than one, new obtuse angle. It follows that an increase in the number of potential triangles is at best matched by an increase in the number of new obtuse angles. Since there is one triangle with no obtuse angles initially, this will remain the case throughout. Thus the task is not possible.

6. Let the sequence be $\{a_n\}$ and let S_n denote the sum of all the terms up to but not including a_n. For $n \geq 2002$, a_n is a divisor of S_n. Hence there exists a positive integer d_n such that $a_n = \frac{S_n}{d_n}$. Then $S_{n+1} = S_n + a_n = \frac{(d_n+1)S_n}{d_n}$. If $d_{n+1} \geq d_n + 1$, then $a_{n+1} \leq \frac{S_n}{d_n} = a_n$, and this contradicts the hypothesis that $\{a_n\}$ is strictly increasing. Hence $\{d_n\}$ is non-decreasing for $n \geq 2002$. However, this sequence cannot maintain a value $k > 1$ indefinitely as otherwise $\{S_n\}$ becomes a geometric progression with common ratio $\frac{k+1}{k}$ starting from some term. However, k and $k+1$ are relatively prime, and we can only divide the first term of the geometric progression by k finitely many times. It follows that $d_n = 1$ eventually.

7. We use induction on the number n of domino pieces in the chain. For $n=1$ and 2, the result holds trivially. Consider the general case where the first number is a. Let the first piece in the initial chain be (a,b) and that in the final chain be (a,c). If $b=c$, we can appeal to the induction hypothesis. Assume therefore that $b \neq c$. Then the piece (a,b) is now further down the chain. If it has been reversed to (b,a), we simply take the sub-chain from (a,c) to (b,a) and reverse it. Then we appeal to the induction hypothesis. Assume therefore that (a,b) has not been reversed. The proof will be complete if we can show that (a,b) can be reversed. In the initial chain, let (d,e) be the first piece which does not appear after (a,b) in the final chain. Let the piece before (d,e) in the initial chain be (f,d). Then this piece appears in the final chain after (a,b), possibly reversed. On the other hand, the piece (d,e) appears in the final chain before (a,b), also possibly reversed. We consider four possible configurations of the final chain, and verify that in each case, (a,b) is reversed.
Case 1. $(a,c),\dots,(d,e),\dots,(a,b),\dots,(f,d),\dots$.
We reverse the sub-chain from (d,e) to (f,d).
Case 2. $(a,c),\dots,(d,e),\dots,(a,b),\dots,(d,f),\dots$.
We reverse the sub-chain from (d,e) to (g,d), where (g,d) is the piece right before (d,f).
Case 3. $(a,c),\dots,(e,d),\dots,(a,b),\dots,(f,d),\dots$.
We reverse the sub-chain from (d,h) to (f,d), where (d,h) is the piece right after (e,d).
Case 4. $(a,c),\dots,(e,d),\dots,(a,b),\dots,(d,f),\dots$.
We reverse the sub-chain from (d,i) to (j,d), where (d,i) is the piece right after (e,d) and (j,d) is the piece right before (d,f).