Seniors
(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

O-Level Paper **Fall 2002.**

1 Let j and m be numbers selected by J and M respectively. Note that $j|2002$; otherwise J would know that $m = 2002 - j$. Also $j \neq 2002$; otherwise $m = 1$ (since $m \neq 0$). So, $j \leq 1001$.

Further, the same is true for m. In addition, M knows that $j \leq 1001$. Therefore, $m = 1001$ (otherwise M would know $j = 2002 : m$).

So, $m = 1001$ is the only possible solution. One can check that it works.

2 Let N be the number of students in the class, M the number of the problems, P the number of passed students, H the number of hard problems. According to definition "a problem is hard" if it has not been solved by at least rN students; where $r = \frac{2}{3}, \frac{3}{4}, \frac{7}{10}$ in (a), (b),(c). Also, according to definition "a student passes" if he solves at least rM problems.

a) *It is possible.* Consider a class consisting of students S_1, S_2, S_3 and set of problems P_1, P_2, P_3. Let S_1 solve P_1 and P_3, S_2 solve P_2 and P_3 and S_3 solved neither P_1 nor P_2. Then S_1, S_2 pass and P_1, P_2 are hard problems.

b) *It is impossible.* Let us write down the results of the test ("+" or "−") into $N \times M$ table.

Let passed students be on the top and hard problems on the left of the table. Let us estimate K_+ and K_-, the numbers of "+" and "−" in the table. First,

$$K_+ \geq (\text{number of " + "got by students who passed}) \geq P \times rM \geq r^2 MN$$

and

$$K_- \geq (\text{number of " − " got for hard problems}) \geq H \times rN \geq r^2 MN.$$

Then $MN = K_+ + K_- \geq 2r^2 MN$ which is impossible for $r = \frac{3}{4}$.

c) *It is impossible.* Arguments of (b) do not work here since $2r^2 \leq 1$. Now we denote by K_+ and K_- the numbers of "+" and "−" in the top-left $P \times H$ sub-table. Then

$$K_+ \geq (\text{minimal number of " + " for hard problems got by students who passed}) \geq P \times \frac{4}{7} H$$

(a student cannot pass if he solves less than $\frac{4}{7}H$ of hard problems even if he solves all the easy problems, the number of which does not exceed $\frac{3}{7}M$). On the other hand,

$$K_- \geq (\text{minimal number of " − " got by students who passed for hard problems}) \geq H \times \frac{4}{7} P.$$

So, $PH = K_+ + K_- \geq \frac{8}{7} PH$ which is impossible.

3 Let us assume that such point B exists (separated from A by each line). Then segment AB intersects all the lines and therefore ray $[BA)$ originated at B has no points of intersection beyond A. Therefore, A belongs to unbounded region.

Now, assume that A belongs to unbounded region. Our region is convex, bounded by two rays and maybe several segments. Note, that these rays are divergent. Therefore, one can draw a ray, originated at A and lying inside of our region. Without any loss of the generality we can assume that this ray is not-parallel to any of the lines; otherwise we can rotate it slightly. Then the opposite ray (originated at A) intersects all the lines and any point B beyond the last point of intersection satisfies the condition.

4 Since function $\cos x$ is a monotone decreasing on $(0, \pi/2)$ we have $(x - y)(\cos x - \cos y) \leq 0$ (equality holds only for $x = y$). Also $(x - z)(\cos z - \cos x) \leq 0$ and $(y - z)(\cos y - \cos z) \leq 0$. Adding these inequalities we get

$$2(x \cos x + y \cos y + z \cos z) \leq (y + z) \cos x + (x + z) \cos y + (y + x) \cos z$$

and therefore

$$3(x \cos x + y \cos y + z \cos z) \leq (x + y + z)(\cos z + \cos y + \cos x)$$

which implies our inequality.

5 Let $\{a_k\}$ be our sequence. Note that $1 \leq a_{k+1} - a_k \leq 9$. Then the segment $[9 \ldots 989, 9 \ldots 999]$ contains a term of our sequence; $a_k = 9 \ldots 99r$. If r is even than a_k is even. If r is odd then a_{k+1} must be odd.