Seniors

(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper **Fall 2002.**

1 *The answer is negative.* It is sufficient to give an example of two numbers which have the same gcd with all the other numbers from 2 to 20,000.

Examples: a) 2^{13} and 2^{14};

b) $19,993$ and $19,997$; both numbers are primes because they have no prime divisors less than 142.

2 Proof by a contradiction. Assume that pentagon has sides ranging from 0.8 to 1.2. To get a pentagon in cross-section of a cube, a plane has to cross five faces, two pairs of which are parallel. Therefore the pentagon has two pairs of parallel sides. Let us consider pentagon $BCDKL$ with $BC \parallel DK$ and $CD \parallel LB$. Then A be a point of intersection of BL and KD (extended). Note that $ABCD$ is a parallelogram. Due to triangle inequality $AL+AK > LK$, then $AB + AD > BL + LK + KD$. So, $BC + CD > BL + LK + KD$. Then even if BC and CD are two longest sides, $BC + CD \leq 2 \cdot 1.2 = 2.4$ while $BL + LK + KD \geq 3 \cdot 0.8 = 2.4$ which is contradiction.

3 Since in N-gon the sum of all angles equals $(N-2) \cdot 180°$, then N-gon is split into $(N-2)$ triangles by $(N-3)$ diagonals, not intersecting inside of N-gon. Side of each white (black) triangle we call white (black); so diagonals are both black and white.

Then, there are at least $(N-3)$ white (black) sides; therefore there are at least $\lceil \frac{1}{3}(N-3) \rceil$ triangles of each color. Let $R(N)$ be the difference in question. Let us consider 3 cases:

a) $N = 3k$. Then there are at least $k-1$ black triangles, at most $2k-1$ white triangles and thus $R(N) \leq k$.

b) $N = 3k+1$. Then there are at least k black triangles, at most $2k-1$ white triangles and thus $R(N) \leq k-1$.

c) $N = 3k+2$. Then there are at least k black triangles, at most $2k$ white triangles and thus $R(N) \leq k$.

Let us prove that all these estimates are sharp and equalities could be reached. For $N = 3, 4, 5$ $(k = 1)$ one can check it easily. For larger N one can construct example by induction by k.

Let us assume that for some k we have corresponding N-gon with the required difference (white triangles are in excess). Then we add a pentagon (2 white and 1 black triangles) to N-gon matching black side of pentagon with the white one of N-gon. Then N increases by 3, k increases by 1 and $R(N)$ increases by 1.

4 Let us start from

Proposition. From any set $\{a_1, \dots, a_n\}$ of n integers one can choose a number or several numbers with their sum divisible by n.

Proof. Let us assume that none of the numbers is divisible by n. Consider numbers $b_1 = a_1$, $b_2 = a_1 + a_2$, …, $b_n = a_1 + a_2 + \ldots + a_n$. If none of them is divisible by n then at least two numbers b_j and b_l ($k < l$) have the same remainders. Then their difference $a_{j+1} + \ldots + a_l$ is divisible by n.

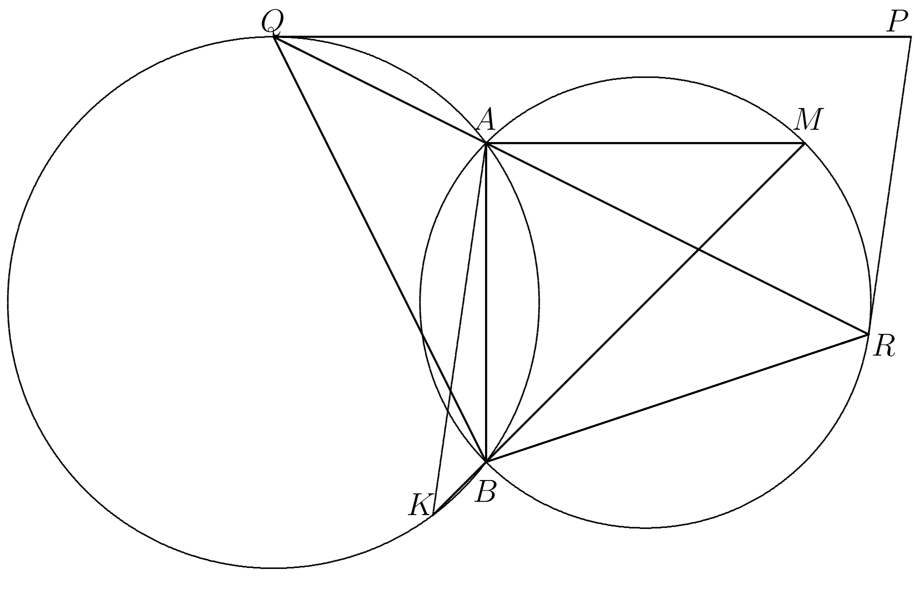
Let us apply an induction by n. If $n = 1$ then only number 1 is written on each card. So, every card by itself forms a required group (with sum 1!).

Assume that a main statement is proven for $(n-1)$, meaning that if the sum of the numbers on all cards is $k \cdot (n-1)!$ then cards could be arranged into k stacks with the sum of the numbers in each stuck equal $(n-1)!$.

Lets call *a supercard* any group of cards with sum $l \cdot n$, $l = 1, \dots, n-1$. We call l *a supercard value*. Any card with number n on it is a supercard of value 1. From the rest of cards with numbers $1, \dots, n-1$ we form supercards by the following procedure: pick any n cards; due to proposition choose several with the sum divisible by n; they form a supercard by definition. This procedure stops when less than n cards are left. However, their sum must be divisible by n (since the total sum and sum on each supercard are divisible by n) meaning that leftovers also form a supercard (sum does not exceed $(n-1)n$).

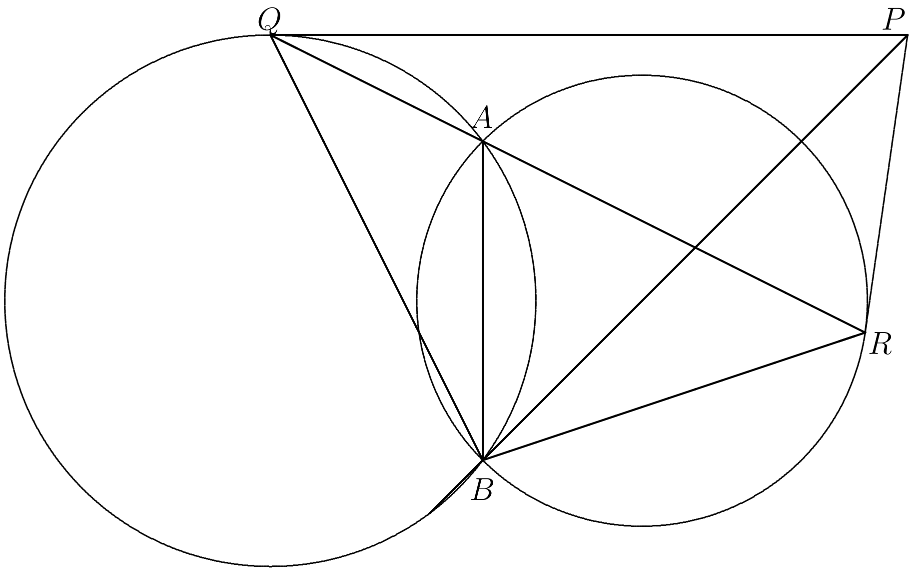
Now we have a pile of supercards with values $1, \dots, n-1$, the total sum of the values equals $(k \cdot n!)/n = k \cdot (n-1)!$. Then according to induction assumption, we can split supercards into k stacks with the sum of the values in each equal $(n-1)!$. Therefore the sum of cards (normal) in each stuck is $(n-1)! \cdot n = n!$.

5 Denote the point of intersection of the two tangents by P.



(a) By Thales' Theorem, $\angle RBM = \angle RAM$. Since AM and QP are parallel, we have $\angle RAM = \angle RQP$. Since QP is tangent to the first circle, $\angle RQP = \angle QBA$. Similarly, $\angle ARP = \angle ABR = \angle ABM + \angle RAM = \angle ABM + \angle RAM = \angle QBM$. By Thales' Theorem, $\angle QAK = \angle QBK$. Hence $\angle QBM = 180° - \angle QBK = 180° - \angle QAK = \angle KAR$. From $\angle ARP = \angle KAR$, we conclude that AK and PR are parallel.

(b) We have $\angle QPR + \angle QBR = \angle QPR + \angle QBA + \angle RBA = \angle QPR + \angle AQP + \angle ARP = 180°$. Hence $BQPR$ is cyclic so that $\angle PBQ = \angle PRQ = \angle MBQ$ from (a). Hence P lies on MB.



6

Proposition 1. If p is prime and a sequence contains an infinite number of multiples of p then it contains all multiples of p.

Proof. Let us assume that for some k our sequence does not contain pk. If $p|a_n$ and $a_{n+1} \neq pk$ then $a_{n+1} < pk$. This could happen only for a finite number of terms multiple of p.

Proposition 2. Our sequence contains all even numbers.

Proof. It is enough to prove that our sequence contains an infinite number of even terms. Assume that it is not the case. Then for some n all terms starting from a_n are odd. Note, that sequence contains an infinite number of terms a_m (with $m \geq n$) such that $a_{m+1} > a_m$. Let $d = \gcd(a_m, a_{m+1})$, d is odd. Note that $a_m + d < a_{m+1}$ and is not coprime with a_m and therefore $a_m + d$ is a term of our sequence. Note that $a_m + d$ is even. Therefore our sequence contains an infinite number of even terms. Contradiction.

Proposition 3. Our sequence contains all odd numbers.

Proof. Let z be the smallest odd number which is skipped in our sequence. Note that the sequence contains all numbers $2kz$. Each such term should be followed by a term which is less than z. This could happen only for a finite number of terms.

7 Solution for $(2k-1)\times(2k-1)$ lattice ($4k^2$ nodes).

For any test a technician chooses a pair of nodes. If the number of tests is less than $2k^2$, at least one node would not be tested. It could happen that this node is isolated but the rest of the wires are intact. So, at least $2k^2$ tests are needed.

Let us numerate the nodes along the main diagonal Δ of the grid from $1,\ldots,2k$. Let us test pairs of nodes $(1,k+1)$, $(2,k+2)$, $\ldots(k,2k)$ plus every pair of nodes which are symmetrical with respect to Δ $(k+k(2k-1)=2k^2)$. Assume that all tests were successful. We need to prove that there is a link between every pair of nodes.

First, we prove that there is a link (connection) between every pair of nodes on Δ. Since nodes 1 and $k+1$ are linked, there exits a path π between them formed by intact wires. Consider a path π' symmetrical to π with respect to Δ. Notice that any node of π' is linked to the symmetrical node of π. Therefore every node of π' is linked to node 1 and therefore all nodes of π' are linked between themselves.

Note that node 2 is either encircled by $\pi\bigcup\pi'$ or belongs to both π and π'. Since nodes 2 and $k+2$ are linked then the intact path between them intersects $\pi\bigcup\pi'$ and therefore both 2 and $k+2$ are linked to 1. Similarly, all other diagonal nodes are linked to 1; therefore all of them are linked.

Now let us consider any non-diagonal node. It is linked with its symmetrical node; the intact path connecting them intersects Δ. This means that any two nodes are linked.