

# Seniors

(Grades 11 and up)

## International Mathematics TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper

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- 1 *The answer is negative.* It is sufficient to give an example of two numbers which have the same gcd with all the other numbers from 2 to 20,000.

Examples: a)  $2^{13}$  and  $2^{14}$ ;

b) 19,993 and 19,997; both numbers are primes because they have no prime divisors less than 142.

- 2 Proof by a contradiction. Assume that pentagon has sides ranging from 0.8 to 1.2. To get a pentagon in cross-section of a cube, a plane has to cross five faces, two pairs of which are parallel. Therefore the pentagon has two pairs of parallel sides. Let us consider pentagon  $BCDKL$  with  $BC \parallel DK$  and  $CD \parallel LB$ . Then  $A$  be a point of intersection of  $BL$  and  $KD$  (extended). Note that  $ABCD$  is a parallelogram. Due to triangle inequality  $AL + AK > LK$ , then  $AB + AD > BL + LK + KD$ . So,  $BC + CD > BL + LK + KD$ . Then even if  $BC$  and  $CD$  are two longest sides,  $BC + CD \leq 2 \cdot 1.2 = 2.4$  while  $BL + LK + KD \geq 3 \cdot 0.8 = 2.4$  which is contradiction.

- 3 Since in  $N$ -gon the sum of all angles equals  $(N - 2) \cdot 180^\circ$ , then  $N$ -gon is split into  $(N - 2)$  triangles by  $(N - 3)$  diagonals, not intersecting inside of  $N$ -gon. Side of each white (black) triangle we call white (black); so diagonals are both black and white.

Then, there are at least  $(N - 3)$  white (black) sides; therefore there are at least  $\lceil \frac{1}{3}(N - 3) \rceil$  triangles of each color. Let  $R(N)$  be the difference in question. Let us consider 3 cases:

- a)  $N = 3k$ . Then there are at least  $k - 1$  black triangles, at most  $2k - 1$  white triangles and thus  $R(N) \leq k$ .
- b)  $N = 3k + 1$ . Then there are at least  $k$  black triangles, at most  $2k - 1$  white triangles and thus  $R(N) \leq k - 1$ .
- c)  $N = 3k + 2$ . Then there are at least  $k$  black triangles, at most  $2k$  white triangles and thus  $R(N) \leq k$ .

Let us prove that all these estimates are sharp and equalities could be reached. For  $N = 3, 4, 5$  ( $k = 1$ ) one can check it easily. For larger  $N$  one can construct example by induction by  $k$ .

Let us assume that for some  $k$  we have corresponding  $N$ -gon with the required difference (white triangles are in excess). Then we add a pentagon (2 white and 1 black triangles) to  $N$ -gon matching black side of pentagon with the white one of  $N$ -gon. Then  $N$  increases by 3,  $k$  increases by 1 and  $R(N)$  increases by 1.

4 Let us start from

*Proposition.* From any set  $\{a_1, \dots, a_n\}$  of  $n$  integers one can choose a number or several numbers with their sum divisible by  $n$ .

*Proof.* Let us assume that none of the numbers is divisible by  $n$ . Consider numbers  $b_1 = a_1$ ,  $b_2 = a_1 + a_2$ ,  $\dots$ ,  $b_n = a_1 + a_2 + \dots + a_n$ . If none of them is divisible by  $n$  then at least two numbers  $b_j$  and  $b_l$  ( $k < l$ ) have the same remainders. Then their difference  $a_{j+1} + \dots + a_l$  is divisible by  $n$ .

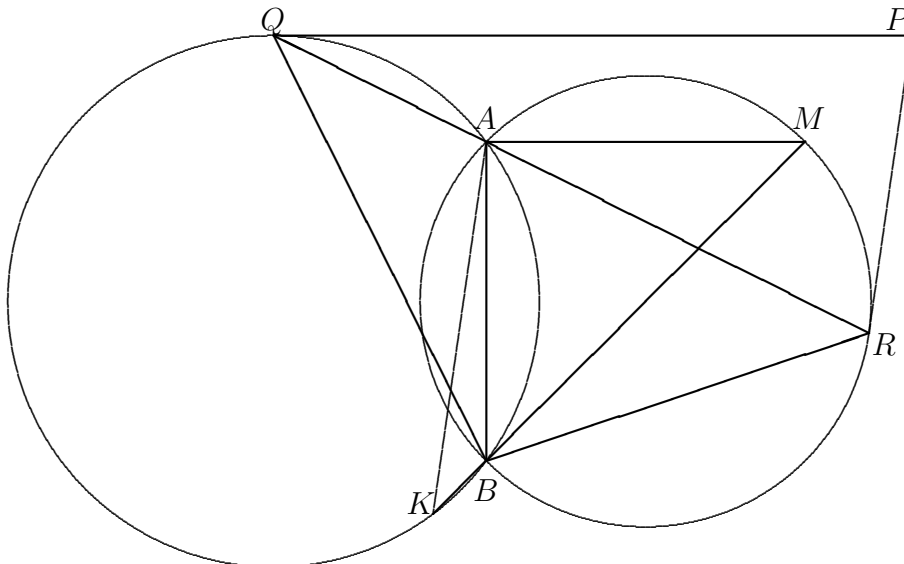
Let us apply an induction by  $n$ . If  $n = 1$  then only number 1 is written on each card. So, every card by itself forms a required group (with sum 1!).

Assume that a main statement is proven for  $(n - 1)$ , meaning that if the sum of the numbers on all cards is  $k \cdot (n - 1)!$  then cards could be arranged into  $k$  stacks with the sum of the numbers in each stuck equal  $(n - 1)!$ .

Lets call a *supercard* any group of cards with sum  $l \cdot n$ ,  $l = 1, \dots, n - 1$ . We call  $l$  a *supercard value*. Any card with number  $n$  on it is a supercard of value 1. From the rest of cards with numbers  $1, \dots, n - 1$  we form supercards by the following procedure: pick any  $n$  cards; due to proposition choose several with the sum divisible by  $n$ ; they form a supercard by definition. This procedure stops when less than  $n$  cards are left. However, their sum must be divisible by  $n$  (since the total sum and sum on each supercard are divisible by  $n$ ) meaning that leftovers also form a supercard (sum does not exceed  $(n - 1)n$ ).

Now we have a pile of supercards with values  $1, \dots, n - 1$ , the total sum of the values equals  $(k \cdot n!)/n = k \cdot (n - 1)!$ . Then according to induction assumption, we can split supercards into  $k$  stacks with the sum of the values in each equal  $(n - 1)!$ . Therefore the sum of cards (normal) in each stuck is  $(n - 1)! \cdot n = n!$ .

5 Denote the point of intersection of the two tangents by  $P$ .





7 Solution for  $(2k - 1) \times (2k - 1)$  lattice ( $4k^2$  nodes).

For any test a technician chooses a pair of nodes. If the number of tests is less than  $2k^2$ , at least one node would not be tested. It could happen that this node is isolated but the rest of the wires are intact. So, at least  $2k^2$  tests are needed.

Let us numerate the nodes along the main diagonal  $\Delta$  of the grid from  $1, \dots, 2k$ . Let us test pairs of nodes  $(1, k + 1), (2, k + 2), \dots, (k, 2k)$  plus every pair of nodes which are symmetrical with respect to  $\Delta$  ( $k + k(2k - 1) = 2k^2$ ). Assume that all tests were successful. We need to prove that there is a link between every pair of nodes.

First, we prove that there is a link (connection) between every pair of nodes on  $\Delta$ . Since nodes 1 and  $k + 1$  are linked, there exists a path  $\pi$  between them formed by intact wires. Consider a path  $\pi'$  symmetrical to  $\pi$  with respect to  $\Delta$ . Notice that any node of  $\pi'$  is linked to the symmetrical node of  $\pi$ . Therefore every node of  $\pi'$  is linked to node 1 and therefore all nodes of  $\pi'$  are linked between themselves.

Note that node 2 is either encircled by  $\pi \cup \pi'$  or belongs to both  $\pi$  and  $\pi'$ . Since nodes 2 and  $k + 2$  are linked then the intact path between them intersects  $\pi \cup \pi'$  and therefore both 2 and  $k + 2$  are linked to 1. Similarly, all other diagonal nodes are linked to 1; therefore all of them are linked.

Now let us consider any non-diagonal node. It is linked with its symmetrical node; the intact path connecting them intersects  $\Delta$ . This means that any two nodes are linked.