Seniors
(Grades 11 and up)

International Mathematics
TOURNAMENT OF THE TOWNS

A-Level Paper **Fall 2002.**

1 [**4**] All the species of plants existing in Russia are catalogued (numbered by integers from 2 to 20 000; one after another, without omissions or repetitions). For any pair of species, the greatest common divisor of their catalogue numbers was calculated and recorded, but the numbers themselves were lost (as the result of a computer error). Is it possible to restore the catalogue number for each specie from that data?

2 [**6**] A cube is cut by a plane so that the cross-section is a pentagon. Prove that the length of one of the sides of the pentagon differs from 1 meter by at least 20 centimeters.

3 [**6**] A convex N-gon is divided by diagonals into triangles so that no two diagonals intersect inside of the polygon. The triangles are painted in black and white so that any two triangles with common side are painted in different colors. For each N, find the maximal difference between the numbers of black and white triangles.

4 [**8**] There is a large pile of cards. On each card one of the numbers $\{1, 2, \ldots, n\}$ is written. It is known that the sum of all numbers of all the cards is equal to $k \cdot n!$ for some integer k. Prove that it is possible to arrange cards into k stacks so that the sum of numbers written on the cards in each stack is equal to $n!$.

5 Two circles intersect at points A and B. Through point B a straight line is drawn, intersecting the first and second circle at points K and M (different from B) respectively. Line ℓ_1 is tangent to the first circle at point Q and parallel to line AM. Line QA intersects the second circle at point R (different from A). Further, line ℓ_2 is tangent to the second circle at point R. Prove that

- **a)** [4] ℓ_2 is parallel to AK;
- **b)** [4] Lines ℓ_1, ℓ_2 and KM have a common point.

6 [**8**] A sequence with first two terms equal 1 and 2 respectively is defined by the following rule: each subsequent term is equal to the smallest positive integer which has not yet occurred in the sequence and is not coprime with the previous term. Prove that all positive integers occur in this sequence.

7 a) [4] A power grid has the shape of a 3×3 lattice with 16 nodes (vertices of the lattice) joined by wires (along the sides of the squares). It may have happened that some of the wires are burned out. In one test technician can choose any pair of nodes and check if electrical current circulates between them (that is, check if there is a chain of intact wires joining the chosen nodes). Technician knows that current will circulate from any node to any other node. What is the least number of tests which is required to demonstrate this?

7 b) [5] The same question for a grid in the shape of a 7×7 lattice (36 nodes).

Keep the problem set.
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