Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

O-Level Paper **Fall 2002.**

1 Consider a triangulation of 2002-gon satisfying the conditions. Triangles which contain at least one side of 2002-gon we call *exterior triangles*. So, our problem is reduced to the following question:

Is it possible to have exactly 1000 exterior triangles? (then we have exactly 1000 triangles which have diagonals for all three sides).

The answer is negative. Really, every exterior triangle contains at most 2 sides of 2002-gon and there should be at least 1001 of them. Contradiction.

2 Let j and m be numbers selected by J and M respectively. Note that $j|2002$; otherwise J would know that $m = 2002 - j$. Also $j \neq 2002$; otherwise $m = 1$ (since $m \neq 0$). So, $j \leq 1001$.

Further, the same is true for m. In addition, M knows that $j \leq 1001$. Therefore, $m = 1001$ (otherwise M would know $j = 2002 : m$).

So, $m = 1001$ is the only possible solution. One can check that it works.

3 Let N be the number of students in the class, M the number of the problems, P the number of passed students, H the number of hard problems. According to definition "a problem is hard" if it has not been solved by at least rN students; where $r = \frac{2}{3}, \frac{3}{4}, \frac{7}{10}$ in (a), (b),(c). Also, according to definition "a student passes" if he solves at least rM problems.

a) *It is possible.* Consider a class consisting of students S_1, S_2, S_3 and set of problems P_1, P_2, P_3. Let S_1 solve P_1 and P_3, S_2 solve P_2 and P_3 and S_3 solved neither P_1 nor P_2. Then S_1, S_2 pass and P_1, P_2 are hard problems.

b) *It is impossible.* Let us write down the results of the test ("+" or "−") into $N \times M$ table.

Let passed students be on the top and hard problems on the left of the table. Let us estimate K_+ and K_-, the numbers of "+" and "−" in the table. First,

$$K_+ \geq (\text{number of " + "got by students who passed}) \geq P \times rM \geq r^2MN$$

and

$$K_- \geq (\text{number of " − " got for hard problems}) \geq H \times rN \geq r^2MN.$$

Then $MN = K_+ + K_- \geq 2r^2MN$ which is impossible for $r = \frac{3}{4}$.

c) *It is impossible.* Arguments of (b) do not work here since $2r^2 \le 1$. Now we denote by K_+ and K_- the numbers of "+" and "−" in the top-left $P \times H$ sub-table. Then

$$K_+ \ge (\text{minimal number of " + " for hard problems got by students who passed}) \ge P\times\frac{4}{7}H$$

(a student cannot pass if he solves less than $\frac{4}{7}H$ of hard problems even if he solves all the easy problems, the number of which does not exceed $\frac{3}{7}M$). On the other hand,

$$K_- \ge (\text{minimal number of " − " got by students who passed for hard problems}) \ge H\times\frac{4}{7}P.$$

So, $PH = K_+ + K_- \ge \frac{8}{7}PH$ which is impossible.

4 The First Player (FP) wins. Let us pair all the cards (numbers): we pair k with $1000 + k$, $k = 1, \dots, 1000$. Also we pair 2001 with 2002. So, in each pair save the last one both cards have the same last digit.

FP starts and picks up 2002. From this moment his strategy is to pick up the other half of the pair chosen by SP. So, eventually SP is forced to pick up 2001. If cards are not gone, then FP takes any card leaving for SP to pick up the other half of the pair. At the end FP has the sum $\equiv 45000 + 2 \equiv 2$ (modulo 10) and SP has the sum $\equiv 1$ (modulo 10).

5 Let us assume that there are straight lines MZ, NY and LX passing through A such that $MN = NL$ and $XY = YZ$ where M, N, L are points on one side of the angle and X, Y, Z are points on the other side. Let us draw a straight line through X parallel to ML; it intersects lines NY and MZ at points N_1 and M_1 respectively. $\triangle AMN$ and $\triangle AM_1N_1$ are similar; so are $\triangle ANL$ and $\triangle XAN_1$. Then $XN_1 = N_1M_1$. Given the assumption $XY = YZ$ we have that lines $N_1Y \parallel M_1Z$, which is impossible since they intersect at A.