Juniors

(Grades up to 10)

International Mathematics
TOURNAMENT OF THE TOWNS: SOLUTIONS

A-Level Paper **Fall 2002.**

1 *Answer: \$ 3002.* First, let us prove that d (the difference in salaries) does not exceed 3002. Let us number employees in clock-wise direction starting from one with the minimal salary. Let n be the employee with the maximal salary. Then 1 and n are separated by $n-2$ employees in clock-wise and $(2002-n)$ counter-clockwise. So $d \leq 3(n-1)$ and $d \leq (2003-n)$. Then $d \leq 3(n-1+2003-n)/2 = 3003$. Note, that $d = 3003$ is only possible if the difference between any two neighbors is exactly 3, which contradicts to assumption that all employees have different salaries.

Let us construct an example with the difference 3002. Let $S(k)$ be a salary of k-th worker. Let $S(1) = 0$, $S(2) = 2$, $S(k) = S(k-1) + 3$ for $k = 3, 4, \ldots, 1002$, $S(1003) = S(1002) - 2$, $S(k) = S(k-1) - 3$ for $k = 1004, \ldots, 2002$. Then $S(1002) - S(1) = 3002$.

2 *The answer is negative.* It is sufficient to give an example of two numbers which have the same gcd with all the other numbers from 2 to 20,000.

Examples: a) 2^{13} and 2^{14};

b) $19,993$ and $19,997$; both numbers are primes because they have no prime divisors less than 142.

3 Let AB be an arc from A to B in clock-wise direction. For any ordered pair of opposite arcs AB and CD we define $d(AB)$ equal to the difference between arc DA and arc BC. Obviously $d(AB)$ is divisible by 50 (because the difference between two opposite arcs is ± 25 and we have 24 pairs).

Now let us switch to next pair of opposite arcs in clock-wise direction. Note that the increment of $d(AB)$ is either 50, or -50, or 0. Also note that $d(CD) = -d(AB)$. Therefore at some moment we reach a pair of opposite arcs with difference 0.

Then corresponding sides of polygon are parallel.

4 Let us encircle $\triangle ABC$. Let K be an intersection point of continuation of BP and encircle. Then $\angle ABK = \angle ACK$ and $\angle CBK = \angle CAK$ (subtended by the same arc). Then $\triangle APC \cong \triangle AKC$ (A-S-A). Therefore $PK \perp AC$. Similarly, we prove that $AP \perp BC$ as well.

5 Since in N-gon the sum of all angles equals $(N-2) \cdot 180°$, then N-gon is split into $(N-2)$ triangles by $(N-3)$ diagonals, not intersecting inside of N-gon. Side of each white (black) triangle we call white (black); so diagonals are both black and white.

Then, there are at least $(N-3)$ white (black) sides; therefore there are at least $\lceil \frac{1}{3}(N-3) \rceil$ triangles of each color. Let $R(N)$ be the difference in question. Let us consider 3 cases:

a) $N = 3k$. Then there are at least $k-1$ black triangles, at most $2k-1$ white triangles and thus $R(N) \leq k$.

b) $N = 3k+1$. Then there are at least k black triangles, at most $2k-1$ white triangles and thus $R(N) \leq k-1$.

c) $N = 3k+2$. Then there are at least k black triangles, at most $2k$ white triangles and thus $R(N) \leq k$.

Let us prove that all these estimates are sharp and equalities could be reached. For $N = 3, 4, 5$ ($k = 1$) one can check it easily. For larger N one can construct example by induction by k.

Let us assume that for some k we have corresponding N-gon with the required difference (white triangles are in excess). Then we add a pentagon (2 white and 1 black triangles) to N-gon matching black side of pentagon with the white one of N-gon. Then N increases by 3, k increases by 1 and $R(N)$ increases by 1.

6 Let us start from

Proposition. From any set $\{a_1, \ldots, a_n\}$ of n integers one can choose a number or several numbers with their sum divisible by n.

Proof. Let us assume that none of the numbers is divisible by n. Consider numbers $b_1 = a_1$, $b_2 = a_1 + a_2$, $\ldots$, $b_n = a_1 + a_2 + \ldots + a_n$. If none of them is divisible by n then at least two numbers b_j and b_l ($k < l$) have the same remainders. Then their difference $a_{j+1} + \ldots + a_l$ is divisible by n.

Let us apply an induction by n. If $n = 1$ then only number 1 is written on each card. So, every card by itself forms a required group (with sum 1!).

Assume that a main statement is proven for $(n-1)$, meaning that if the sum of the numbers on all cards is $k \cdot (n-1)!$ then cards could be arranged into k stacks with the sum of the numbers in each stuck equal $(n-1)!$.

Lets call *a supercard* any group of cards with sum $l \cdot n$, $l = 1, \ldots, n-1$. We call l *a supercard value*. Any card with number n on it is a supercard of value 1. From the rest of cards with numbers $1, \ldots, n-1$ we form supercards by the following procedure: pick any n cards; due to proposition choose several with the sum divisible by n; they form a supercard by definition. This procedure stops when less than n cards are left. However, their sum must be divisible by n (since the total sum and sum on each supercard are divisible by n) meaning that leftovers also form a supercard (sum does not exceed $(n-1)n$).

Now we have a pile of supercards with values $1, \ldots, n-1$, the total sum of the values equals $(k \cdot n!)/n = k \cdot (n-1)!$. Then according to induction assumption, we can split supercards into k stacks with the sum of the values in each equal $(n-1)!$. Therefore the sum of cards (normal) in each stuck is $(n-1)! \cdot n = n!$.

7 Solution for $(2k-1) \times (2k-1)$ lattice ($4k^2$ nodes).

For any test a technician chooses a pair of nodes. If the number of tests is less than $2k^2$, at least one node would not be tested. It could happen that this node is isolated but the rest of the wires are intact. So, at least $2k^2$ tests are needed.

Let us numerate the nodes along the main diagonal Δ of the grid from $1, \dots, 2k$. Let us test pairs of nodes $(1, k+1)$, $(2, k+2)$, $\dots (k, 2k)$ plus every pair of nodes which are symmetrical with respect to Δ $(k + k(2k-1) = 2k^2)$. Assume that all tests were successful. We need to prove that there is a link between every pair of nodes.

First, we prove that there is a link (connection) between every pair of nodes on Δ. Since nodes 1 and $k+1$ are linked, there exits a path π between them formed by intact wires. Consider a path π' symmetrical to π with respect to Δ. Notice that any node of π' is linked to the symmetrical node of π. Therefore every node of π' is linked to node 1 and therefore all nodes of π' are linked between themselves.

Note that node 2 is either encircled by $\pi \bigcup \pi'$ or belongs to both π and π'. Since nodes 2 and $k+2$ are linked then the intact path between them intersects $\pi \bigcup \pi'$ and therefore both 2 and $k+2$ are linked to 1. Similarly, all other diagonal nodes are linked to 1; therefore all of them are linked.

Now let us consider any non-diagonal node. It is linked with its symmetrical node; the intact path connecting them intersects Δ. This means that any two nodes are linked.