SOLUTIONS OF TOURNAMENT OF TOWNS

Spring 2001, Level A, Senior (grades 11-OAC)

Problem 1 [3] Find at least one polynomial $P(x)$ of degree 2001 such that $P(x) + P(1-x) = 1$ holds for all real numbers x.

SOLUTION. It is easy to see that polynomial

$$P(x) = (1-x)^{2001} - x^{2001} + \frac{1}{2}$$

satisfies identity $P(x) + P(1-x) = 1$.

Problem 2 [5] At the end of the school year it became clear that for any arbitrarily chosen group of no less than 5 students, 80% of the marks "F" received by this group were given to no more than 20% of the students in the group. Prove that at least 3/4 of all "F" marks were given to the same student.

SOLUTION. Let us arrange all the students in the school according to the number of "F"marks they received. So, $F_1 \geq F_2 \geq \cdots \geq F_n$ where F_j is the number of "F" received by j-th student, $1 \leq j \leq n$, $F_j \geq 0$ and $\sum_{j=1}^{n} F_j = F$ where F is a total number of "F" marks.

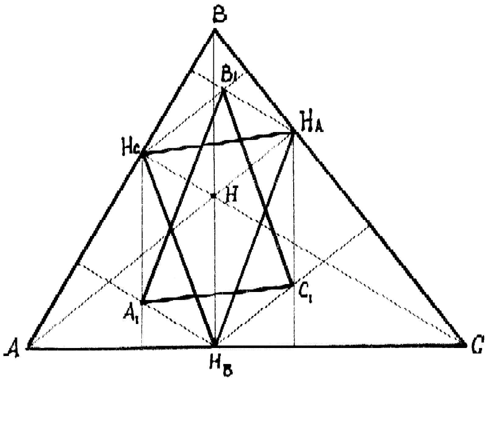
Now let us consider the first five students. According to the condition, one student (who has to be on top of the list) got at least 80% of "F" marks received by this group which leaves no more than 20% of "F" marks remaining for the other four students. So, $F_2 + F_3 + F_4 + F_5 \leq \frac{1}{4}F_4$ and we have an estimate $F_2 \leq \frac{1}{4}F_1$. Considering students from k-th to $k+4$-th ($k+4 \leq n$) we conclude that $F_{k+1} \leq \frac{1}{4}F_k$ which implies that $F_{k+1} \leq \frac{1}{4^k}F_1$ ($k \leq n-5$) and $F_{n-3} + F_{n-2} + F_{n-1} + F_n \leq \frac{1}{4}F_{n-4}$.

Now we have

$$F = F_1 + F_2 + \cdots + F_{n-4} + (F_{n-3} + \cdots + F_n) \leq$$
$$F_1 + \frac{1}{4}F_1 + \frac{1}{4^2}F_1 + \cdots + \frac{1}{4^{n-5}}F_1 + \frac{1}{4^{n-4}}F_1 < \sum_{k=0}^{\infty} \frac{1}{4^k}F_1 = \frac{F_1}{1-\frac{1}{4}} = \frac{4}{3}F_1;$$

Therefore $F_1 > \frac{3}{4}F$.

Problem 3 [5] Let AH_A, BH_B and CH_C be the altitudes of triangle $\triangle ABC$. Prove that the triangle whose vertices are the intersection points of the altitudes of $\triangle AH_BH_C$, $\triangle BH_AH_C$ and $\triangle CH_AH_B$ is congruent to $\triangle H_AH_BH_C$.



SOLUTION. Let us notice that $H_CA_1H_BH$ and $HH_AC_1H_B$ are parallelograms (HH_A and H_BC_1 are perpendicular to BC; H_CA_1, HH_B and H_AC_1 are perpendicular to AC; HH_C and H_BA_1 are perpendicular to AB). Therefore $H_CA_1 = H_AC_1$ and since they are parallel we conclude that $H_CH_AC_1A_1$ is parallelogram, thus $H_CH_A = A_1C_1$. In a similar way we can prove that $H_BH_A = A_1B_1$ and $H_CH_B = B_1C_1$. Therefore $\triangle H_CH_AH_B \cong \triangle A_1B_1C_1$.

Problem 4 [5] There are two matrices A and B of size $m \times n$ each filled only by "0"s and "1"s. It is given that along any row or column its elements do not decrease (from left to right and from top to bottom). It is also given that the numbers of "1"s in both matrices are equal and for any $k = 1, \ldots, m$ the sum of the elements in the top k rows of the matrix A is no less than that of the matrix B. Prove for any $l = 1, \ldots, n$ the sum of the elements in left l columns of the matrix A is no greater than that of the matrix B.

Solution. Let us denote the elements of matrices A and B by a_{ij} and b_{ij} respectively where a_{ij} and b_{ij} are equal to 0 or 1. Notice that if $a_{ij} = 1$ then $a_{i'j'} = 1$ for all $i' \geq i, j' \geq j$ and the same is true for b_{ij}.

Let us assume that for some l

(*) The sum of the elements in left l columns of the matrix A is no greater than that of the matrix B.

Let us consider the minimal l with this property. Let k be the number of "1"s in l-th column of matrix A. Notice that k exceeds the number of "1"s in the same column of matrix B, otherwise we would not have (*). Note that the l-th column and the p-th row ($p = m - k + 1$) divide the matrices into four parts defined by relations:

$P_1 : 1 \leq i \leq p - 1$ and $1 \leq j \leq l$;
$P_2 : 1 \leq i \leq p - 1$ and $l + 1 \leq j \leq n$;
$P_3 : p \leq i \leq m$ and $1 \leq j \leq l$;
$P_4 : p \leq i \leq m$ and $l + 1 \leq j \leq m$;

Let $N_A, N_B, N_{A_k}, N_{B_k}$ be the number of "1"s in matrices A, B and their parts.

Now we will compare the number of "1"s in all parts.

In P_1 we have $N_{A_1} = N_{B_1} = 0$.
In $P_1 \cup P_2$ we have $N_{A_1} + N_{A_2} \geq N_{B_1} + N_{B_2}$ according to the condition of the problem.
In $P_1 \cup P_3$ we have $N_{A_1} + N_{A_3} \geq N_{B_1} + N_{B_3}$ due to our assumption.
In P_4 we have $N_{A_4} \geq N_{B_4}$ because this part of A consists of "1"s only.

Therefore, $N_A > N_B$ which contradicts the condition of the problem.

Problem 5 In a chess tournament, every participant played with each other exactly once, receiving 1 point for a win, 1/2 for a draw and 0 for a loss.

(a) [4] Is it possible that for every player P, the sum of points of the players who were beaten by P is greater than the sum of points of the players who beat P?
(b) [4] Is it possible that for every player P, the first sum is less than the second one?

Solution. Let ε_{ij} be result of the game between i-th and j-th players:

$$\varepsilon_{ij} = \begin{cases} 1 & \text{if } i\text{-th player wins,} \\ -1 & j\text{-th player wins,} \\ 0 & \text{if they have a draw or } i = j. \end{cases}$$

Then (a) asks if it is possible that

$$\sum_{j=1}^{n} \varepsilon_{ij} X_j > 0 \quad \forall i$$

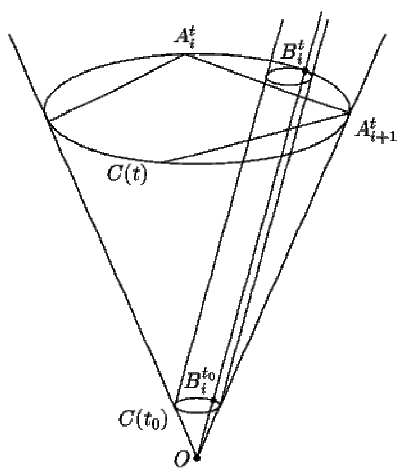
where X_j is a score of j-th player. Multiplying these inequalities by X_i and summing up for all i we conclude that

$$\sum_{i,j=1}^{n} \varepsilon_{ij} X_i X_j > 0.$$

This is impossible since the left-hand expression is 0 because $\varepsilon_{ij} = -\varepsilon_{ji}$ for all i, j. Part (b) is considered in a similar way.

Problem 6 [8] Prove that there exist 2001 convex polyhedra such that any three of them do not have any common points but any two of them touch each other (i.e., have at least one common boundary point but no common inner points).

Solution. Let us set $N = 2001$ and describe construction of N convex polyhedra satisfying the conditions of the problem.



Let us consider an infinite straight circular cone K with the vertex at the origin and an axis directed along OZ. Let $C(t)$ be a circle with the center $O(t) = (0,0,t)$ obtained by an intersection of K and a plane $\{z = t\}$. Let us consider a regular N-gon inscribed in $C(1)$, with vertices A_i. Let B_i be the middle points of arcs A_iA_{i+1}, $i = 1,\dots,N$. Let us denote by $A_i^t, B_i^t \in C(t)$ the points of intersection of generating lines OA_i, OB_i with $\{z = t\}$, $t > 0, 1 \le i \le N$. Now we need a following

Lemma. *For any $t_0 > 0$ and any $i, 1 \le i \le N$ there exists $T > t_0$ such that for all $t \ge T$ the parallel translation of $C(t_0)$ by the vector $\overrightarrow{B_i^{t_0} B_i^t}$ lies inside the segment $S_i(t) = A_i^t B_i^t A_{i+1}^t$ (bounded by an arc and a straight segment).*

The proof is based on the fact that the distance from B_i^t to $A_i^t A_{i+1}^t$ is proportional to t.

Now we construct the polyhedra satisfying the conditions of the problem by induction. Let us start with any $t_1 > 0$. We may choose any convex polygon M_1 inside a circle $C(t_1)$ and form an infinite "up" prism P_1 with the base M_1 and lateral edges parallel to OB_1.

Now suppose that we already defined numbers $0 < t_1 < \cdots < t_{n-1}$, constructed convex polygons $M_1, \dots, M_{n-1}$ contained in circles $C(t_1), \dots, C(t_{n-1})$ and formed infinite prims $P_1, \dots, P_{n-1}$ with bases $M_1, \dots, M_{n-1}$ and lateral edges parallel to $OB_1, \dots, OB_{n-1}$ ($n < N$), satisfying the conditions of the problem.

According to the lemma there exists $t_n > t_{n-1}$ such that $M_{n-1}(t_n)$ lies inside the segment $S_{n-1}(t_n)$ (and all the previous polygons would remain in their segments).

Now we need to define M_n. It has to touch each of the previous prisms. In order to find the points of tangency we make a parallel translation of every segment $A_i^{t_n} A_{i+1}^{t_n}$ until it touches the polygon $M_i(t_n)$ ($1 \le i \le n-1$). Now connecting points of tangency we get a convex polygon $M_n(t_n)$ (we choose any point of tangency if there are many; for $n = 2, 3$ we add extra vertices). For future arguments we introduce translated lines $\ell_i(t_n)$; these lines separate M_i and M_n.

Now we form an infinite "up" prism P_n with base $M_n(t_n)$ and lateral edges parallel to OB_n. Construction ends when we use the last segment; at this moment we cut the prisms by a plane $\{z = T > t_N\}$.

Now we need to check that these prisms satisfy conditions of the problem. It is enough to prove that P_n intersects P_i $(i < n)$ only in the plane $\{z = t_n\}$. Assume that this is not true. Then there exists a common point $R \in \{z = t\}$ of these prisms, $t > t_n$.

Let us draw straight lines parallel to OB_n and OB_i through R. These lines intersect plane $\{z = t_n\}$ at points $R_n^{t_n}$ and $R_i^{t_n}$ respectively. Notice that $R_n^{t_n} \in M_n(t_n)$ and $R_i^{t_n} \in M_i(t_n)$. Also notice that vectors $\overrightarrow{R_i^{t_n} R_n^{t_n}}$ and $\overrightarrow{B_i^{t_n} B_n^{t_n}}$ have opposite directions and are not equal to 0.

This cannot be true since $R_i^{t_n}$ and $B_i^{t_n}$ lie on one side of $\ell_i(t_n)$ and $R_n^{t_n}$ and $B_n^{t_n}$ lie on the other side. This contradiction completes the proof.

Problem 7 Several boxes are arranged in a circle. Each box may be empty or may contain one or several chips. A move consists of taking all the chips from some box and distributing them one by one into subsequent boxes clockwise starting from the next box in the clockwise direction.

(a) [4] Suppose that on each move (except for the first one) one must take the chips from the box where the last chip was placed on the previous move. Prove that after several moves the initial distribution of the chips among the boxes will reappear.

(b) [4] Now, suppose that in each move one can take the chips from any box. Is it true that for every initial distribution of the chips you can get any possible distribution?

Solution. (a) Let the *state* of the system described in the problem be defined by the *distribution* of chips between boxes and the number of a box from which we move. Notice that the sequence of states is uniquely defined going forward. Moreover, it is uniquely defined going backwards. Really, if we start with the box where we put the last chip, and go in the counter-clockwise direction, collecting one chip from each box until we get to an empty box, then put all the collected chips in this box, we restore the previous state.

Let us notice that the number of different states is finite. Therefore, we can conclude that the sequence of states is cyclic. Therefore the initial state will repeat itself.

(b) Now the sequence of states is not uniquely defined. Let us mark some box (M). Let I be a state when all chips are collected in M.

Lemma 1. *State I can be obtained from any state A.*

Proof. Let us consider a box (M_{-1}) next to a marked one (M) in a counter-clockwise direction. Starting with that box we would increase the number of chips in M and empty box M_{-1}. Now we do the same with M_{-2}. When it is empty we will return to M_{-1} again. By doing so, each time we increase the number of chips in M and the number of empty boxes until all chips are collected in M. □

Lemma 2. *Let A and B be two states, such that B can be obtained from A. Then A can be obtained from B.*

Proof. Let us consider a case when we get B from A in one step. Starting with the box where the last chip was put and continuing according to part (a) rules, we will come eventually to A. The general case can be considered by induction.

Now it is easy to get the final result. Really, from any two states A and B we can get the state I. Therefore, we can get B from I and thus from A. □