International Mathematics

22nd Tournament of Towns

Spring 2001, Ordinary Level

Solutions

Junior (Grades 7, 8, 9 and 10)

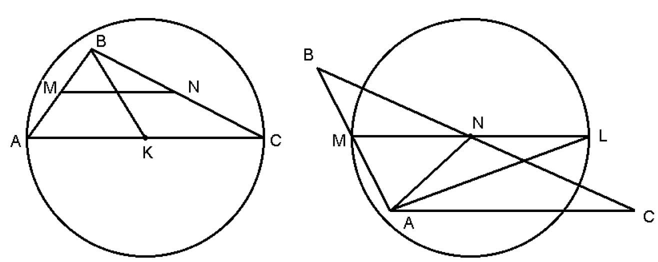
1. [3] The natural number n can be replaced by ab if $a+b=n$, where a and b are natural numbers. Can the number 2001 be obtained from 22 after a sequence of such replacements?

Solution. Yes, it can. In fact, there are infinitely many ways of obtaining 2001 from 22. First note that $n=(n-1)+1$, so from n we can obtain $n-1$. Now it is enough to get any number larger than 2001 and then descend to 2001 one by one. For example we can do: $22=11+11 \rightarrow 121=60+61 \rightarrow 3660 \rightarrow 3659 \rightarrow \cdots \rightarrow 2001$.

2. [4] One of the midlines of triangle $\triangle ABC$ is longer than one of its medians. Prove that the triangle has an obtuse angle.

Solution. Let M and N be the midpoints of AB and BC, respectively. Assume first that midline MN is longer than median BK. Let us draw a circle centered at K with radius $AK=KC=MN$, so AC is its diameter. Since BK is shorter than the radius MN, point B lies inside the circle. Therefore, angle $\angle ABC$ is obtuse.

Now assume MN is longer than one of the other two medians, say $|MN|>|AN|$. Let us draw a circle centered at N with radius MN. Let ML be its diameter. Again since AN is shorter than the radius MN, point A lies inside the circle. Therefore angle $\angle MAL$ is obtuse and hence BAC is obtuse.



3. [4] Twenty kilograms of cheese are on sale in a grocery store. Several customers are lined up to buy this cheese. After a while, having sold the demanded portion of cheese to the next customer, the salesgirl calculates the average weight of the portions of cheese already sold and declares the number of customers for whom there is exactly enough cheese if each customer will buy a portion of cheese of weight exactly equal to the average weight of the previous purchases. Could it happen that the salesgirl can declare every time a customer has made their purchase, that there just enough cheese for the next 10 customers? If so, how much cheese will be left in the store after the first 10 customers have made their purchases? (The average weight of a series of purchases is the total weight of the cheese sold divided by the number of purchases.)

Solution. Let a_k be the amount of cheese kth customer bought. Then $(a_1 + \cdots + a_k)/k$ is the average amount of cheese sold to the first k customers and $20 - (a_1 + \cdots + a_k)$ is the remaining amount. The salesgirl declares that this would be enough for exactly 10 more customers if they buy the average amount each. This means that

$$20 - (a_1 + \cdots + a_k) = 10\,\frac{a_1 + \cdots + a_k}{k},$$

or

$$a_1 + \cdots + a_k = \frac{20k}{k+10}.$$

Since this holds for every k we also have

$$a_1 + \cdots + a_{k-1} = \frac{20(k-1)}{k+9}.$$

Subtracting one from the other we get

$$a_k = \frac{200}{(k+9)(k+10)} > 0.$$

Therefore if kth customer buys exactly this amount of cheese, then the condition of the problem will be met.

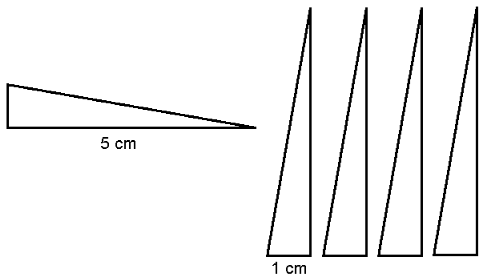
Under the above assumption, we get

$$a_1 + \cdots + a_{10} = \frac{20 \cdot 10}{10 + 10} = 10.$$

Thus there will be $20 - 10 = 10$ kg of cheese left.

4 a. [2] There are 5 identical paper triangles on the table. Each can be moved in any direction parallel to itself (i.e., without rotating it). Is it true that then any one of them can be covered by the 4 others?

Solution. No. In the following example first triangle cannot be covered by the other four.



4 b.[3] There are 5 identical equilateral paper triangles on the table. Each can be moved in any direction parallel to itself. Prove that any one of them can be covered by the 4 others in this way.

Solution. Each equilateral triangle contains an inscribed disc inside it. Each such disc can cover an equilateral triangle of twice smaller side. Let us divide one triangle into 4 equal equilateral triangles of twice smaller side, by drawing its three midlines. Then let's cover these four small triangles by the discs inside the other four triangles. We can do this by moving the triangles parallel to themselves. Thus, any triangle can be covered by the other four in this way.

5. [5] On a square board divided into 15×15 little squares there are 15 rooks that do not attack each other. Then each rook makes one move like that of a knight. Prove that after this is done a pair of rooks will necessarily attack each other.
Solution. Let us number the rows and the columns of the board by numbers from 1 to 15. Then every square is represented by a pair of numbers (a, b), where a, b are between 1 and 15. Let (a_k, b_k) represents the square on which kth rook is placed. Since at the beginning the rooks do not attack each other, in every row (column) there is exactly one rook. Therefore the numbers $a_1, \ldots, a_{15}$ are all numbers from 1 to 15. The same for $b_1, \ldots, b_{15}$. We conclude that if the rooks do not attack each other then the sum $S = a_1 + \cdots + a_{15} + b_1 + \cdots + b_{15} = 15 \cdot 16$ is even.

We will show now that after each rook makes a move of a knight the sum S becomes odd. Indeed, when kth rook makes a move of a knight a_k changes by 1 and b_k changes by 2 or a_k changes by 2 and b_k changes by 1. Thus, $a_k + b_k$ changes by either 1 or 3. Since we have an odd number of rooks the sum S becomes odd after all rooks have made their moves. But this means that a pair of rooks will attack each other, otherwise, as we proved above, the sum S would be even.