International Mathematics
TOURNAMENT OF THE TOWNS

Senior A-Level Paper **Fall 2001.**

1. On the plane is a triangle with red vertices and a triangle with blue vertices. O is a point inside both triangles such that the distance from O to any red vertex is less than the distance from O to any blue vertex. Can the three red vertices and the three blue vertices all lie on the same circle?

2. Do there exist positive integers $a_1 < a_2 < \cdots < a_{100}$ such that for $2 \leq k \leq 100$, the least common multiple of a_{k-1} and a_k is greater than the least common multiple of a_k and a_{k+1}?

3. An 8×8 array consists of the numbers 1, 2, ..., 64. Consecutive numbers are adjacent along a row or a column. What is the minimum value of the sum of the numbers along a diagonal?

4. Let F_1 be an arbitrary convex quadrilateral. For $k \geq 2$, F_k is obtained by cutting F_{k-1} into two pieces along one of its diagonals, flipping one piece over and then glueing them back together along the same diagonal. What is the maximum number of non-congruent quadrilaterals in the sequence $\{F_k\}$?

5. Let a and d be positive integers. For any positive integer n, the number $a + nd$ contains a block of consecutive digits which constitute the number n. Prove that d is a power of 10.

6. In a row are 23 boxes such that for $1 \leq k \leq 23$, there is a box containing exactly k balls. In one move, we can double the number of balls in any box by taking balls from another box which has more. Is it always possible to end up with exactly k balls in the k-th box for $1 \leq k \leq 23$?

7. The vertices of a triangle have coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3). For any integers h and k, not both 0, the triangle whose vertices have coordinates (x_1+h, y_1+k), (x_2+h, y_1+k) and (x_3+h, y_3+k) has no common interior points with the original triangle.

   (a) Is it possible for the area of this triangle to be greater than $\frac{1}{2}$?

   (b) What is the maximum area of this triangle?

Note: The problems are worth 4, 5, 6, 6, 7, 7 and 3+6 points respectively.