International Mathematics
TOURNAMENT OF THE TOWNS

Junior O-Level Paper **Fall 2001.**

1. In the quadrilateral $ABCD$, AD is parallel to BC. K is a point on AB. Draw the line through A parallel to KC and the line through B parallel to KD. Prove that these two lines intersect at some point on CD.

2. Clara computed the product of the first n positive integers and Valerie computed the product of the first m even positive integers, where $m \geq 2$. They got the same answer. Prove that one of them had made a mistake.

3. Kolya is told that two of his four coins are fake. He knows that all real coins have the same weight, all fake coins have the same weight, and the weight of a real coin is greater than that of a fake coin. Can Kolya decide whether he indeed has exactly two fake coins by using a balance twice?

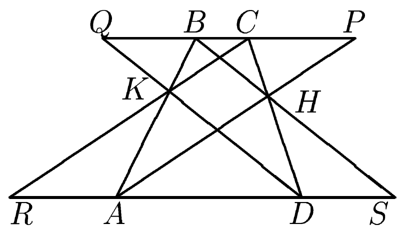
4. On an east-west shipping lane are ten ships sailing individually. The first five from the west are sailing eastwards while the other five ships are sailing westwards. They sail at the same constant speed at all times. Whenever two ships meets, each turns around and sails in the opposite direction. When all ships have returned to port, how many meetings of two ships have taken place?

5. On the plane is a set of at least four points. If any one point from this set is removed, the resulting set has an axis of symmetry. Is it necessarily true that the whole set also has an axis of symmetry?

Note: Each problem is worth 4 points.

Solutions to Junior O-Level Fall 2001

1. Let CK cut the extension of DA at R and DK cut the extension of CB at Q. Let the line through A parallel to KC cut CD at H and the extension of BC at P. Then $APCR$ is a parallelogram, so that $AR = CP$. We have to prove that BH is parallel to KC. Let the extension of BH cut the extension of AD at S. Now triangles BKQ and AKD are similar, as are triangles BKC and AKR. Hence $\frac{BQ}{AD} = \frac{BK}{AK} = \frac{BC}{AR}$. Similarly, $\frac{DS}{BC} = \frac{AD}{CP}$. Hence $BQ = \frac{BC \cdot AD}{AR} = \frac{BC \cdot AD}{CP} = DS$. It follows that $BQDS$ is also a parallelogram, so that BH is indeed parallel to KD.



2. Suppose $n! = 2^m m!$ for some $m \geq 2$). We must have $n > 3$ so that both $n!$ and $m!$ are divisible by 3. In each product, every third factor is a multiple of 3. In order for both products to be divisible by the same power of 3, $n!$ can have at most two more terms than $m!$. If $n = m + 1 = 2^m$, we have $m = 1$. If $n = m + 2$, $(m + 1)(m + 2) = 2^m$ leads to $m = 0$. Both contradicts $m \geq 2$. Hence either Clara or Valerie had made a mistake.

3. Kolya can decide as follows. Label the coins A, B, C and D. In the first weighing, put A and B on one side and C and D on the other. Suppose $A + B = C + D$. In the second weighing, put A on one side and B on the other. If $A = B$, then we have $A = B = C = D$. If $A \neq B$, exactly one of A and B is fake, and exactly one of C and D is fake.

   Suppose $A + B \neq C + D$. In the second weighing, put A and C on one side and B and D on the other. If $A + C = B + D$, then the number of fake coins is even but not 0 or 4. If $A + C \neq B + D$, the number of fake coins is odd.

4. Let us consider what happens when two ships meet. Each continues where the other would have gone. Since we are interested in the total number of meetings rather than the numbers of meetings for individual ships, we may pretend that the ships just sail on. Since there are 5 ships from each side, the total number of meetings is $5 \times 5 = 25$.

5. Let ABC be a triangle with $AB = AC$ and $\angle CAB = 36°$. Let D be a point on AC such that $BC = BD$. Then $\angle BDC = \angle BCD = \angle ABC = 72°$ so that $\angle ABD = \angle DBC = 36°$. Hence $BD = AD$. Consider the set $\{A, B, C, D\}$. It does not have an axis of symmetry. If A is removed, we have $BC = BD$. If B is removed, A, C and D are collinear. If C is removed, we have $AD = BD$. If D is removed, we have $AB = AC$. Each subset of three points has an axis of symmetry.