International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2001.**

1. Do there exist positive integers $a_1 < a_2 < \cdots < a_{100}$ such that for $2 \leq k \leq 100$, the greatest common divisor of a_{k-1} and a_k is greater than the greatest common divisor of a_k and a_{k+1}?

2. Let $n \geq 3$ be an integer. A circle is divided into $2n$ arcs by $2n$ points. Each arc has one of three possible lengths, and no two adjacent arcs have the same length. The $2n$ points are coloured alternately red and blue. Prove that the n-gon with red vertices and the n-gon with blue vertices have the same perimeter and the same area.

3. Let $n \geq 3$ be an integer. Each row in an $(n-2) \times n$ array consists of the numbers 1, 2, ..., n in some order, and the numbers in each column are all different. Prove that this array can be expanded into an $n \times n$ array such that each row and each column consists of the numbers 1, 2, ..., n.

4. Let $n \geq 2$ be an integer. A regular $(2n+1)$-gon is divided into $2n-1$ triangles by diagonals which do not meet except at the vertices. Prove that at least three of these triangles are isosceles.

5. Alex places a rook on any square of an empty 8×8 chessboard. Then he places additional rooks one rook at a time, each attacking an odd number of rooks which are already on the board. A rook attacks to the left, to the right, above and below, and only the first rook in each direction. What is the maximum number of rooks Alex can place on the chessboard?

6. Several numbers are written in a row. In each move, Robert chooses any two adjacent numbers in which the one on the left is greater than the one on the right, doubles each of them and then switches them around. Prove that Robert can make only a finite number of such moves.

7. It is given that 2^{333} is a 101-digit number whose first digit is 1. How many of the numbers 2^k, $1 \leq k \leq 332$, have first digit 4?

Note: The problems are worth 4, 5, 5, 5, 6, 8 and 8 points respectively.

Solutions to Junior A-Level Fall 2001

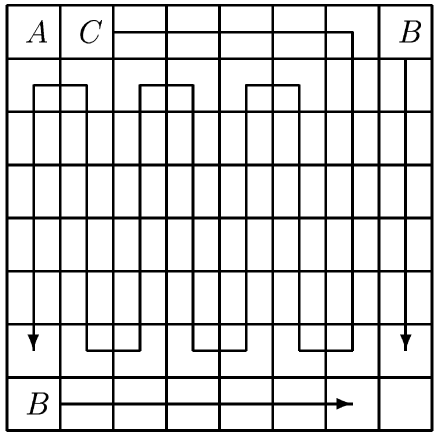
1. For $1 \leq k \leq 100$, let $a_k = 2^{99}+2^{98}+\cdots+2^{100-k}$. Then $a_1 < a_2 < \cdots < a_{100}$. For $2 \leq k \leq 100$, the difference between a_{k-1} and a_k is 2^{100-k}. Since it divides both a_{k-1} and a_k, 2^{100-k} is in fact their greatest common divisor. Similarly, the greatest common divisor of a_k and a_{k+1} is $2^{100-(k+1)}$, which is less than 2^{100-k}. Thus there exist 100 positive integers with the desired properties.

2. Let a, b and c be the three arc lengths. Let there be x arcs of length a, y arcs of length b and z arcs of length c. Then $x+y+z=2n$. Each side of the n-gon with red vertices is subtended by an arc of length $b+c$, $c+a$ or $a+b$. Of these n arcs, x of them contains an arc of length a, so that the number of arcs of length $b+c$ is $n-x$. Similarly, the number of arcs of length $c+a$ is $n-y$, and the number of arcs of length $a+b$ is $n-z$. Exactly the same thing can be said about the n-gon with blue vertices. Hence the two polygons have the same perimeter. By the same reason, the area of the part of the circle outside the n-gon with red vertices is the same as that of the n-gon with blue vertices. It follows that the two polygons also have the same area.

3. Each column is missing two of the numbers, and each number is missed by exactly two columns. Construct a graph with n vertices representing the numbers. Two vertices are joined by an edge if the numbers they represent are both missed by the same column, so that there are exactly n edges. Moreover, each vertex has degree 2. This means that the graph is a union of cycles, including degenerate cycles of length 2. In each cycle, we orient the edges so that they are all directed clockwise. Then each vertex has in-degree 1 and out-degree 1. For each column in the expanded square, locate the directed edge joining vertices representing the numbers missing from this column. Putting the number represented by the initial vertex in the $(n-1)$-st row and the number represented by the terminal vertexin the n-th row. Clearly, all the numbers in each column are distinct. Since each vertex has in-degree 1 and out-degree 1, each number appears exactly once in the $(n-1)$-st row and exactly once in the n-th row.

4. For $2 \leq k \leq n$, consider a path consisting of k consecutive edges of the regular $(2n+1)$-gon. The diagonal joining the endpoints of this path is said to have *span* k. Note that a diagonal of span 2 cuts off an isosceles triangle whose equal sides are sides of the original polygon. If in our triangulation, there is a triangle formed by three diagonals. This divides the remaining parts of the original polygon into three pieces, each of which must contain a diagonal of span 2. It follows that we will have at least three isosceles triangles. The only other case is that each triangle in the triangulation shares a side with the original polygon. Thus the triangles form a sequence such that each shares a diagonal with its neighbours. The first and the last triangles in this sequence are cut of by diagonals of span 2, and are isosceles. The diagonals shared by neighbouring triangles increase in span to n from both directions. Since there are $2n-2$ diagonals, there are two diagonals of span n, which forms the third isosceles triangle with a side of the original polygon.

5. After three corner squares have been occupied, a rook at the fourth corner square will always be attacking two existing rooks. Hence at least one corner square must be empty, so that the maximum number of rooks that can be placed is sixty-three. The diagram below shows how Alex can place as many as sixty-three rooks in three stages labelled A, B and C.



6. Let the numbers be a_1, $a_2, \ldots$, a_n, and we maintain these labels even though the values of the numbers change with the doubling. We claim that two numbers can exchange places at most once. It will then follow that Robert must stop after at most $\binom{n}{2}$ moves. Assuming to the contrary that there are two numbers which exchange places at least twice. Consider such a pair of exchanges that occur the closest together. Let the numbers be a_i and a_j such that $a_i > a_j$ during the first exchange. Then $a_i < a_j$ during the second exchange. In between, a_j must have grown more than a_i. In any exchange involving a_i or a_j with a third number a_k, it must come between them after their first exchange and get out before their second exchange. If a_k gets in through one and goes out through the other, it has no effect on the relative size of a_i and a_j. The only way a_j can outgrow a_i is for it to exchange twice with some a_k in between its two exchanges with a_i. However, this contradicts the assumption that the two exchanges between a_i and a_j are the closest together. This justifies the claim.

7. The number of digits is non-decreasing along the sequence $\{2^k : 1 \leq k \leq 332\}$. Clearly, this number cannot increase by more than 1 at a time. Every time an increase occurs, the new power of 2 must have 1 as its first digit. Since 2^{333} is an 101-digit number, there are exactly 99 numbers in our sequence whose first digit is 1. When the number of digits does not change, the first digit changes in one of the following sequences: 1-2-4-8, 1-2-4-9, 1-2-5, 1-3-6 or 1-3-7. Now the 99 numbers above divide our sequence into 100 blocks, each of length 2 or 3. Let there be x blocks of length 2 and y blocks of length 3. Then $x + y = 100$ while $2x + 3y = 232$, which yield $x = 67$ and $y = 33$. Now each block of length 2 does not contain any number whose first digit is 4, while each block of length 3 contains exactly one number whose first digit is 4. It follows the exactly 33 numbers in our sequence whose first digit is 4.