International Mathematics
TOURNAMENT OF THE TOWNS

Junior A-Level Paper **Fall 2001.**

1. Do there exist positive integers $a_1 < a_2 < \cdots < a_{100}$ such that for $2 \leq k \leq 100$, the greatest common divisor of a_{k-1} and a_k is greater than the greatest common divisor of a_k and a_{k+1}?

2. Let $n \geq 3$ be an integer. A circle is divided into $2n$ arcs by $2n$ points. Each arc has one of three possible lengths, and no two adjacent arcs have the same length. The $2n$ points are coloured alternately red and blue. Prove that the n-gon with red vertices and the n-gon with blue vertices have the same perimeter and the same area.

3. Let $n \geq 3$ be an integer. Each row in an $(n-2) \times n$ array consists of the numbers 1, 2, ..., n in some order, and the numbers in each column are all different. Prove that this array can be expanded into an $n \times n$ array such that each row and each column consists of the numbers 1, 2, ..., n.

4. Let $n \geq 2$ be an integer. A regular $(2n+1)$-gon is divided into $2n-1$ triangles by diagonals which do not meet except at the vertices. Prove that at least three of these triangles are isosceles.

5. Alex places a rook on any square of an empty 8×8 chessboard. Then he places additional rooks one rook at a time, each attacking an odd number of rooks which are already on the board. A rook attacks to the left, to the right, above and below, and only the first rook in each direction. What is the maximum number of rooks Alex can place on the chessboard?

6. Several numbers are written in a row. In each move, Robert chooses any two adjacent numbers in which the one on the left is greater than the one on the right, doubles each of them and then switches them around. Prove that Robert can make only a finite number of such moves.

7. It is given that 2^{333} is a 101-digit number whose first digit is 1. How many of the numbers 2^k, $1 \leq k \leq 332$, have first digit 4?

Note: The problems are worth 4, 5, 5, 5, 6, 8 and 8 points respectively.