Processes and Operations

The *Process* is used when one needs to construct the necessary object. Iterations, Mathematical Induction and Method of Descent are examples of processes.

Problems

1. There are several gas stations along the Big Circle Road. The total supply of gas at these stations is enough to circle the road. Prove that a vehicle with an empty tank can start from some station and circle the road (filling up a tank at gas stations on the way).

2. In the City every road is one-way road and any two roads intersect at most once. One of the roads (Circular Road) completely encircles the City and any other road starts and ends on the Circular Road. Roads split the city into blocks. Prove that at least one block can be circled.

3. Each cell of a $m \times n$ table is filled with a number. It is allowed to flip simultaneously the signs of all numbers in any row or a column. Prove that applying this operation several times one can always get a table with non-negative sums of the numbers in every row and column.

4. Each member of the Parliament has no more than three foes among the other members. Prove that the Parliament can be splited into two fractions so that each member has no more than one foe in his fractions. (Hint: Split parliament into two fractions (somehow). Choose a member who has at least two foes in his fraction and relocate him into the other fraction. Continue the process . It will stop eventually due to the fact that each time the total number of foes in the same fraction decreases).

5. There are n bushes in a row. A hare hides under one of them. A hunter cannot see it and shoots at any bush of his choice. The hare is hit if it happened to be under this bush; otherwise, after the shot the hare runs under a neighboring bush. Is there a strategy that guarantees to hit the hare?

   (Hint: Assume that initially the hair was hiding under a bush with an odd number. Think of a strategy (process) that eliminates this possibility. Then eliminate a possibility that the hair was hiding under a bush with an even number).

6. A circle is inscribed into scalene triangle. A triangle is constructed by connecting the points of tangency. A circle is inscribed into it and so on. Prove that there is no pair of similar triangles in this sequence of triangles.

   (Hint: Look at the pattern of angles in a sequence of triangles).

7. Every digit of a binary number is either 0 or 1. It is allowed to exchange any fragment of “10” to the fragment “0001”. Prove that this process cannot continue forever.

   (Hint: use induction on the number of “1” in a number).

8. Several dots are marked on the plane. Some of them are connected by segments. If a pair of the segments intersects then it can be replaced by another pair of the segments which has the same ends and does not intersect. Can this process continue forever?

(Hint: notice, that each operation results in decreasing of the total sum of the lengths of segments).

9. n pirates divide a pile of treasures between themselves. Each pirate wants to be sure that he gets no less than $1/n$ of the pile. Find a fair way to split the pile between them so that no one can blame anyone(except, maybe, himself). Pirates opinions on size and value of the piles could be different.

10. Each of n identical jars is filled with a paint to $(n-1)/n$ of its volume. No two jars contain the same kind of the paint. It is allowed to pour out any amount of paint from any jar to another jar. Can one get the same mixture in all jars? Paint is not disposable and there is no other empty vessels.

11. (Tournament of Towns 2005, Fall, A-Level, Juniors) There are 100 pots each containing varying amounts of jam, no more than 1/10-th of the total. Each day, exactly 10 pots are to be chosen, and from each pot, the same amount of jam is eaten. Prove that it is possible to eat all the jam in a finite number of days.