Preparation Session 2
October 15, 2006

Review of Methods

1 Proof by contradiction

Suppose that we need to prove some statement A. If by assuming that the opposite statement is true we get a contradiction, then A is true.

Problems

1. Prove that set of all natural numbers (positive integers) is infinite.
2. 5 children together picked up 9 mushrooms. Prove that at least two children picked up the same number of mushrooms.
3. Prove that if n divides $(n-1)!+1$ then n is a prime.
4. Can one cut a convex 12-gon into 9 triangles?
5. Prove that any convex quadrilateral is completely covered by union of four circles, constructed on its sides as on diameters.

2 Extreme case

Sometimes in process of solution it is crucial to consider extreme objects: the largest number, the closest point or a vertex, a degenerate circle or any other extreme case.

Problems

1. Numbers are placed on a chessboard. It is given that any number is equal to an arithmetical mean of its neighbors. Prove that all the numbers are the same.
2. Prove that any polyhedron has at least two faces with the same number of sides.
3. A traveller went from a city A to a city B that is furthest away from A. Then he went from city B to a city C which is furthest away from B and so on. Prove that if C and A are different cities then the traveller will never return to A . The distances between the cities are different.
4. Distinct integers are placed in each square of a chessboard. Prove that there are two adjacent squares (by side) with the numbers in them different by at least 5.
5. Consider a convex polygon and interior point P in it. Let us drop the perpendiculars from P to each side of the polygon (or its extension). Prove that at least one perpendicular meets a corresponding side.

3 Invariants

Suppose that set of some operations is applied to the object. *Invariant* is a value (characteristic of the object) that does not change under given set of operations. Examples of invariants: parity, coloring , residues, value of function, etc. If invariant separates initial and final states of the object (that is, the values of invariant at initial and final states are different) then transformation of the object from initial to the final state by these operations is not possible. However, if invariant is preserved then it does not imply that the transformation in mention is always possible: could be either case. (One have to either to search for another invariant or to produce an example that shows that transformation is possible).

Problems

1. Seven "0" and one "1" are placed at the vertices of a cube. It is allowed to add 1 to the numbers on both ends of any edge. Is it possible to make all numbers equal by applying this operation several times? Is it possible to make all numbers divisible by 3?

2. On a miracle palm tree grow apples and pears, 13 of each. It is allowed to pick up either one or two fruits at the same time. If one picks up one fruit, the same fruit grows instantly. If one picks up two apples or two pears, one pear grows instead. If one picks up one apple and one pear, one apple grows. Could it happen that in the end only one fruit is left? What it would be? Can it happen that in the end no fruit is left?

3. On a chess board "camel" moves by (1,3) rule: it moves 1 square right or left and then 3 square up or down (or 1 square up or down and 3 squares left or right). Is it possible to move camel from a square to a neighboring one?

4. There are three numbers. The following operation is allowed: any two numbers a, b are replaced by $(a+b)/\sqrt{2}, (a-b)/\sqrt{2}$. Is it possible to get $(1, \sqrt{2}, 1+\sqrt{2})$ from $(2, \sqrt{2}, 1/\sqrt{2})$ applying the operation several times ?

4 Looking for a relative problem

If a problem is hard, try to find and solve a simpler problem, related to it. This can give a clue to the original problem. Following advices may be helpful:

- Consider a special (simpler) case and then generalize the idea.
- Break a problem down into the smaller ones.
- Generalize a problem. For example, you can replace a given number by a variable.
- Reduce a problem to a simpler one.

Problems

1. One corner cell of $n \times n$ table is filled with "+" while the rest cells are filled with "$-$". One can choose any row or a column and flip all the signs in it. Can one eventually get a table filled with pluses?

2. One can easily cut $3 \times 3 \times 3$-cube into 27 smaller cubes by 6 cuts. Can one reduce the number of cuts if one is allowed to move pieces between cuts?

3. Solve the equation $(x^2 + x - 3)^2 + 2x^2 + 2x - 5 = 0$.

4. Compare two fractions: 31415926/2345678 **?** 31415927/2345679

5. Construct a tangent to two given circles. HINT: Replace one circle by a point.

5 Pigeon Hole Principle

In a simple way Pigeon hole principle states: If there are $(n + 1)$ pigeons in n holes then there is (at least) one hole with (at least) two pigeons in it.
GENERAL STATEMENT: If there are n pigeons in k holes then there is a hole with at least $\lceil n/k \rceil$ pigeons in it and a hole with at most $\lfloor n/k \rfloor$ pigeons in it .
Although the principle sounds simple (proof by contradiction), it takes practice to learn which objects to consider as pigeons and which as holes.

Problems

1. Is it possible to create a 7×7 table with each entry either $-1, 0$ or 1, so that sums of the numbers in each row, column and each of two diagonals are distinct.

2. There are 15 cities in a country. Each city is directly connected by a road to at least 7 others cities. The roads do not intersect except at the cities. Prove that one can reach any city from any other city along the roads.

3. There are 20 children in a kindergarden. Any two children have the same granddad. Prove that one of the granddads has at least 14 grandchildren, attending the kindergarden.

4. Prove that out of 10 positive integers, none of which is divisible by 10, one can find

   (a) two numbers whose difference is divisible by 10;

   (b) several numbers whose sum is divisible by 10.

6 Going backwards

Backwards search – Starting from the end and going backwards we subsequently define unknown value(s) on previous steps.

1. Greek Gods used to play the following game. All of them got cups with the same amount of wine. If two Gods have the same amount of wine then one of them pours all his wine into a cup of the other. This procedure is repeated. One day it happened that all wine was collected into Zeus's cup. Prove that the number of Gods is a power of 2.

2. Three boys divide 48 stamps between themselves. First Alex gives Ben and Chris the number of stamps that each of them already has. Then Ben gives Alex and Chris the number of stamps that each of them has at the moment. Finally, Chris gives Alex and Ben the number of stamps that each of them has at the moment. In result, they all have the same number of stamps. How many stamps each of the boys had at the beginning?

Backwards search in games – Starting from the end, in succession of descents are defined the winning and loosing positions for each player. Position is declared a winning if there is a move from it to a position previously defined as a loosing position. On the other hand, position is declared a loosing if any move from it leads to a position previously defined as a winning position.

1. There are 10 nuts in a pile. Two players in turns pick up one or two nuts at the time. The player who takes the last nut wins. Which of the players has a winning strategy?

7 Counting in Two Ways

Counting in two ways is applied in following cases:

1. to obtain equation (one needs to express in two ways an unknown value)
2. to obtain estimate (one needs to estimate some value in two ways)
3. to obtain contradiction (one may consider some value in two different ways or apply two ways of arguments).

Problems

1. Evaluate the sum $1 + 3 + 3^2 + 3^3 + \cdots + 3^n$.
2. A knight is placed on a chessboard one square of which is cut out. Could it happen that the knight visited each square of this chessboard exactly ones and returned to its initial position?
3. Can one fill down the entries of 5×5 table with some numbers so that the sum of the numbers in each row is positive while the sum of the numbers in each column is negative?
4. Can one fill down the entries of 3×3 table with some numbers so that the product of the numbers in each row is positive while the product of the numbers in each column is negative? The same question for 4×4 table.
5. On the straight line there is a finite colony of bacteries. Every moment bacteries which have neither left neighbour on the distance 1 nor right neighbour on the distance $\sqrt{2}$ will die. New bacteries are not born. Does there exist a colony which lives forever?

8 Correspondence

Sometimes to compare the number of elements in two (finite) sets it is sufficient to establish a one-to-one correspondence between the elements; if it can be done then the number of elements in both sets is the same. If we can establish a one-to-one correspondence between one set and only part of the second set, then the number of elements of the second set is greater.

1. One of n chosen points on a circle is marked with red colour. Consider all possible convex polygons with vertices at chosen points. Which set is greater: the set of polygons that contain red point or the set of polygons that does not?

2. Two boys toss a coin, one tosses 10 times, while the other tosses 11 times. Who of the boys, the first or the second, has greater chances to get more heads?

3. 20 students winners of Tournament of the Towns could not attend the award ceremony in time and came one week later to pick up their book prizes. The books with names tag attached to them were left at front desk of Math Department.First came a crazy 6-grader. Not paying attention to the tags he took one book and left with it. Every student who came later took his/her book if there was one ; otherwise, picked up any book at random.What are chances that the last student picks up a book that was supposed for him?

Correspondence in Games

Sometimes one can find a reciprocal move that leads to a win (provided by symmetry or by pairing or by complement of a set).

1. Two players A and B in turns place coins (cents) on an empty place of a circle table. The player who can not place a coin (not enough place) loses. Which of the players has a winning strategy, the first or the second?

2. Two chips are placed on the leftmost cell of $1 \times n$ stripe. Two players in turns move one of the chips on any number of cells to the right. Player who can not make a move, loses. Which of the players has a winning strategy, the first or the second?

3. There are 2 piles of candies, m and n. Two children, Anna and Boris in turns pick up any number of candies from one of the piles. The child who takes the last candy wins. Anna starts first. Who of the children has a winning strategy?

9 Induction

Principle of Mathematical Induction

If it is known that
(1) some statement is true for $n = 1$

(2) assumption that statement is true for n implies that the statement is true for (n+1)
then the statement is true for all positive integers

Principle of Mathematical Induction, modification

If it is known that
(1) some statement is true for $n = 1$

(2) assumption that statement is true for all positive integers k, $1 \leq k \leq n$ implies that the statement is true for (n+1)
then the statement is true for all positive integers

Problems

1. Prove the following identity: $1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$
2. Prove that if $(x + 1/x)$ is integer then $(x^n + 1/x^n)$ is also integer for any positive integer n.
3. Let S be a set that contains n elements. Prove, that the set of all subsets S contains 2^n elements.
4. N straight lines in general positions divide a plane into several regions. Find the number of regions. Straight lines are said to be in general position if any pair of lines intersects, and no three lines can go through the same point.
5. A traveller came to a country inn. He had no money but he had a chain made of gold rings. Host of the inn agreed to take gold rings as a payment, each day one more ring. To accomplish the task traveller cut up n rings . Find the maximal number of days (number of rings in the chain) the traveller can pay for his stay?

10 Estimate-Example problems

In an *estimate-example problems*, one is asked to find the smallest (the largest) value of M that satisfies a given property. The problem is tackled in two steps:

- first, one must establish an *estimate* for M
- second, to find an *example* where M *is reached*. Both parts are equally important; however, they may greatly vary in difficulty.

Problems

1. Several flies (distinct points) are sitting on the surface of a cube. Find the minimal number of flies given that no two faces contain the same number of flies
2. An ant crawls on the surface of a cube going from vertex to vertex either by edge or by face diagonal. It is not allowed to intersect the path or to visit the same vertex twice. Find the maximal length of the path from one vertex to the opposite vertex of the cube.
3. Each term of a sequence of natural numbers is obtained from the previous term by adding to it its largest digit. What is the maximal number of successive odd terms in such a sequence?

   (Tournament of the Towns 2003, O-Level, Spring Round)
4. A chess piece moves as follows: it can jump 8 or 9 squares either vertically or horizontally. It is not allowed to visit the same square twice. At most, how many squares can this piece visit on a 15×15 board (it can start from any square)?

   (Tournament of the Towns 2005, O-Level, Fall Round)
5. 25 checkers are placed on 25 leftmost squares of $1 \times N$ board. A checker can either move to the empty adjacent square to its right or jump over an adjacent right checker to the next square if it empty.moves to the left are not allowed. Find minimal N such that all the checkers could be placed in the row of 25 squares but in the reverse order.

   (Tournament of the Towns 2003, O-Level, Fall Round)