How To Solve Problems. V

In a simple way Pigeon Hole Principle states:

If there are $(n+1)$ pigeons in n holes then there is a hole (at least one) with two (at least) pigeons in it.

General statement:

If there are n pigeons in k holes then there is a hole with at least $\lceil n/k \rceil$ pigeons in it and a hole with at most $\lfloor n/k \rfloor$ pigeons in it.

Although the principle sounds simple, it takes practice to learn which objects to consider as pigeons and which as holes.

Problems.

1. Is it possible to create a 7×7 table with each entry either $-1, 0$ or 1, so that sums of the numbers in each row, column and each of two diagonals are all distinct?

2. There are 50 boulders weighing $370 + 2k$, $k = 0, \ldots, 49$ respectively. Is it possible to carry them out by three 7-tons trucks?

3. There are 15 cities in a country. Each city is directly connected by a road to at least 7 other cities. The roads do not intersect except at the cities. Prove that one can reach any city from any other city along the roads.

4. (a) Do there exist four distinct positive integers, such that sum of any three of them is a prime number?

   (b) Do there exist five distinct positive integers, such that sum of any three of them is a prime number?

5. Prove that out of any 10 positive integers, none of which is divisible by 10, one can find

   (a) two numbers with the difference divisible by 10;

   (b) several numbers with the sum divisible by 10.

6. There are 20 children in a kindergarden. Any two children have a common granddad. Prove that one of the granddads has at least 14 grandchildren attending the kindergarden.

7. There are 101 marked points on a plane, not all of them are collinear. Straight lines are drawn through all pairs of the points. Prove that there is a point (among marked) such that at least 11 lines intersect in it.

8. Rectangle 20×30 is split into 600 unit squares. Can one draw a straight line which intersects (at interior points) 50 unit squares?

9. Each of 9 given straight lines split a given square into two trapezoids with area ratio 2:3. Prove that at least three of the straight lines intersect at the same point.