3 Math Club February 18, 2001

3.1 Warm-up problems

Problem 1. There is a pile of candies on a table. The first boy takes 1/10 of candies. The second boy takes 1/10 what is left plus 1/10 candies that the first has. The third boy takes 1/10 what is left plus 1/10 that two boys have, and so on until nothing is left. How many boys are there? Who gets the most of candies?

Problem 2. There is a deck of 100 cards. It is known that for any 50 of them there are 30 cards that for any 20 of them there are 10 cards that for any 5 of them there are at least 3 red cards. At the same time for any 80 cards from initial deck there are 20 cards that for any 10 of them there are at least 2 blue cards. How many red and blue cards are in the deck ?

How to solve problems?
Invariants

Invariant is a characteristic of an object which does not change if some operations are applied to this object. If invariant separates two objects then it is impossible to transform one object to another by these operations. Examples of invariants are parity, coloring and residues.

Problem 1. The numbers are arranged into $m \times n$-table in such a way that the sum of the numbers in any row or a column is equal to 1. Prove that $m = n$.

Problem 2. There are 7 glasses on a table in the bottom-up position. It is allowed to turn over any 4 glasses at a time. Is it possible to reach a position in which all glasses are bottom-down?

Problem 3. Seven "0" and one "1" are put into vertices of a cube. It is allowed to add "1" to the numbers on both ends of any edge. Is it possible to make all numbers are equal applying this operation several times? Is it possible to make all numbers divisible by 3?

Problem 4. On 10×10-chessboard "camel" moves by $(1, 3)$ rule: it moves 1 square right or left and then 3 squares up or down (or 1 square up or down and 3 squares left or right). Is it possible to move camel from a square to a neighbouring one?

Problem 5. Prove that one cannot tile

(a) a regular chessboard by fifteen type (a) and one type (b) figures.
(b) 10×10 chessboard by type (c) figures.
(c) 102×102 chessboard by type (a) figures.

Problem 6. The rectangular is tiled by 1×4 and 2×2 tiles. Then tiles were removed and one 1×4 tile was replaced by 2×2 tile. Is it possible to tile this rectangular by the new set?

Problem 7. There are 1001 pebbles in a pile. It is allowed to remove one pebble from any pile containing more than one pebble and in the same time split one of the piles into two. Is it possible repeating this operation to make all piles containing three pebbles each?

Problem 8. Can one cut a convex 17-gon into 14 triangles?

Problem 9. Can one cut a circle into several parts which could be rearranged into square (cuts are straight lines and circular arcs)?

Problem 10. There are three numbers. The following operation is allowed with any two of them: a, b are replaced by $(a+b)/\sqrt{2}, (a-b)/\sqrt{2}$. Is it possible to get $(1, \sqrt{2}, 1+\sqrt{2})$ from $(2, \sqrt{2}, 1/\sqrt{2})$?

Problem 11. There are two checkers on a straight line: white on the left of the black. Two operations are allowed: (1) insertion of two checkers of the same color next to each other in any place; (2) removal of two checkers of the same color next to each other in any place. Is it possible to get exactly two checkers left: black on the left of the white.

Problem 12. Rainbow Island is populated by chameleons of three colors: 13 of them are blue, 15 are green and 17 are red. If two of chameleons of different colors meet they both change their colors to the third one. Is it possible that at some moment all chameleons will be of the same color?

Problem 13. 44 trees grow in a circle. Initially on each tree there is a squirrel. From time to time any two squirrels jump on neighboring trees: one in clock-wise direction and the other in counter-clock-wise direction. Is it possible for all squirrels to gather on the same tree?

Problem 14. During talks knights from two hostile clans were sitting at round table. It is known that the number of knights who had a friend on his right was equal to the number of knights who had a foe on his right. Prove that the total number of knights is divisible by 4.

Problem 15. Bananas and pine-apples grow on a magic apple-tree. It is allowed to take off two fruits at the same time. If one takes two bananas or two pine-apples then exactly one new pine-apple grows. If one takes one banana and one pine-apple then exactly one new banana grows. If there were b bananas and p pine-apples at the beginning and later only one fruit left, what is it?

Problem 17. The circle is divided into six sectors which are marked by $1, 0, 1, 0, 0, 0$ in clock-wise direction. It is allowed to increase by 1 numbers in two neighbouring sectors. Is it possible repeating this operation to make all numbers equal?

Problem 16. Numbers $1, 2, \ldots, n$ are written in a row. It is allowed to permute any two numbers. Is it possible to return to initial position after 2001 permutations?

Problem 18. Long ago the game "15" was very popular. Prove that one cannot permute "15" and "14" leaving others at their places.

Prove the same statement for 24 checkers in 5×5 square.

3.2 Olympiads archive

Problem 1 (Tournament of Towns). There are 20 pupils in some class. Any two of them have a common grandfather. Prove that there is a grandfather who has at least 14 grandchildren in this class.

Problem 2 (Tournament of Towns). Is it possible that in the second dance every boy danced with prettier or smarter girl than in the first dance, and at least one boy danced with the girl who was both prettier and smarter? (the numbers of boys and girls are equal)

Problem 3. A computer can execute only one operation: to calculate the arithmetic mean of two integers. Three integers are given: $0, m, n$ such that $0 < m < n$, $\gcd(m, n) = 1$. Prove that one can get (a) 1; (b) any integer from 1 to n.

Problem 4 (Tournament of Towns). The angle A of an isoseles triangle ABC ($AB = AC$) is equal to α. Let D be a point on the side AB such that

$AD = AB/n$, $n > 2$. Find the sum $\sum_{k=1}^{n-1} DP_kA$, where P_k are the points dividing the side BC into n equal parts

Problem 5 (St. Petersburg Mathematical Olympiad). Several people are sitting at a round table. Each of of them has some nuts. At a certain signal, each person passes some of his nuts to the person sitting to his right: if he has an even number of nuts, he passes half of them. Otherwise, he takes one nut from the plate in the center and gives half to his neighbour on the right. Prove that at some moment everybody will have the same number of nuts.

Problem 6 (Tournament of Towns). Ten people are sitting at a round table. They have 100 nuts among them. At a certain signal, each person passes some of his nuts to the person sitting to his right: if he has an even number of nuts, he passes half of them. Otherwise, he passes one nut plus half of the remainder. Prove that at some moment everybody will have the same number of nuts.

Problem 7 (Tournament of Towns). A labyrinth" is an 8×8 chessboard with barriers between some pairs of neighboring squares. If a rook can traverse the entire board without crossing any barriers, the labyrinth is "good": otherwise, it is "bad". Are there more good labyrinths or more bad labyrinths?

Problem 8 (St. Petersburg Mathematical Olympiad) Positive integers are written in the cells of the rectangular table. One is allowed to double all the numbers of any row and also to substract 1 from every number of any column. Prove that applying these two operations many times one can get a table filled by zeroes.

Problem 9 (MathBattle) Point E is given on the diameter AC of the circle. Draw a chord BD through E in such a way that an area of the quadrilateral $ABCD$ is maximal.

Problem 10 (Tournament of Towns) Let P be a point inside the triangle $\triangle ABC$ with $AB = BC$, $\angle ABC = 80°$, $\angle PAC = 40°$, and $\angle ACP = 30°$. Find $\angle BPC$.

Problem 11 (Tournament of Towns). The intelligence quotient (IQ) of a country is defined as the average IQ of its entire population. It is assumed that the total population and individual IQ's remain constant throughout.

(a) (i) A group of people from country A has emigrated to country B. Show that it can happen that as a result, the IQ's of both countries have increased.

(ii) After this, a group of people from B, which may include immigrants from A, emigrates to A. Can it happen that the IQ's of both countries will increase again?

(b) A group of people from country A has emigrated to country B, and a group of people from B has emigrated to country C. It is known that as a result, the IQ's of all three countries have increased. After this, a group of people from C emigrates to B and a group of people from B emigrates to A. Can it happen that the IQ's of all three countries will increase again?

Problem 12 (Tournament of Towns) You are given a balance and one copy of each ten weights of $1, 2, 4, 8, 16, 32, 64, 128, 256$ and 512 grams. An object weighing M grams, where M is a positive integer, may be balanced in different ways by placing various combinations of the given weights on either pans of the balance.

(a) Prove that no object may be balanced in more than 89 ways.

(b) Find a value of M such that an object weighing M grams can be balanced in 89 ways.

Problem 13 (Russian Olympiad). The Council of Wizards is tested in the following way: The King lines the wizards up in a line and places on the head of each of them either a white hat or a blue hat or a red hat. Each wizard sees the colors of hats of the people standing in front of him, but he neither sees the color of his hat nor the colors of hats of the people standing behind. Every minute some of the wizards must announce one of the three colors (it is allowed to speak out just once). After completion of this procedure the King executes all the wizards who failed to guess the right color of their hats. Prior to this ceremony all 100 members have agreed to minimize the number of executions. How many of them are definitely secure against the punishment?

Problem 14 (Tournament of Towns). (a) A magician draws 5 cards from a 52-card deck at random, looks at them, and arranges them in a row from left to right, with one card, not necessarily the first. facing down and the others facing up. The assistant "guesses" the card which is face down. Prove that the performers can agree on a system which makes this work every time.

(b) The second trick is the same except that the magician arranges only four cards, facing up, but keeps the fifth card hidden. Can the performers still agree on a system which enables the assistant to "guess," the hidden card?