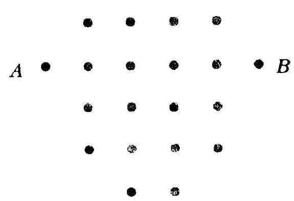
3 Math Club January 21, 2001

3.1 Quiz "Lets have some fun!"

Problem 1. *Musketeering swordplay*. At the royal fencing competition in France, the first four places were taken by Athos, Porthos, Aramis, and D'Artagnan. The sum of the places taken by Athos, Porthos, and D'Artagnan was 6; the sum of the places taken by Porthos and Aramis was also 6. What was the place taken by each of the musketeers if Porthos ranked higher than Athos?

Problem 2. There are 20 nails in the board, with the distances between neighbours 1 cm. Pull the 19-cm thread from A to B in such a way that it passes through all the nails.



Problem 3. There are 30 pikes in a pond. Pikes could eat one another. Pike is *sated* if it ate 3 other pikes (sated or hungry). What is the maximal number of pikes which could be sated (sated pike eaten later also counts)

Problem 4. Two girls in turn tear off petals of cauliflower. In each turn one can tear off one or two neighbouring petals. A girl who tears off the last petal is lucky. Which of the girls has a winning strategy?

Problem 5. There are 100 dwarves with the masses $1, 2, \ldots, 100$ kgs. What is the least number of teams in which they could be divided given that no team contains two dwarves one of whom is twice as heavy as the second one?

3.2 How to solve problems?

3.2.1 Extreme case

Sometimes in a solution of a problem it is crucial to consider extreme objects: the largest number, the closest point or vertex, a degenerate circle, any other limit case.

The minimal counterexample method is a combination of *the extreme case* method and a *a proof by the contradiction*: Assuming that some statement A is not true, there exists a minimal (in some sense) counterexample. If somehow we can "decrease" it, then we get a contradiction.

Problem 1. Numbers are placed on a chessboard. It is given that any number is equal to an arithmetical mean of its neighbours by a side. Prove that all numbers are equal.

Problem 2. Prove that any polyhedron has at least two faces with the same number of sides.

Problem 3. Prove that for any $n \geq 2$, $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not an integer.

Problem 4. A traveller went from his city A to the city B of his country, furthest away from A. Then from B he went to the city C which is furthest away from B and so on. Prove that if C and A are different cities then the traveller will never return to A (all distances between cities of this country are different).

Problem 5. Let us consider a point P inside a convex polygon and drop the perpendiculars from P to each side or its extension. Prove that at least one of the perpendiculars meets a corresponding side.

Problem 6. A 100-head dragon wants to place each of its heads in such a way that each head will be between two others (you can consider heads as points in space). Is this possible?

Problem 7. Distinct integers are placed on the squares of the chessboard. Prove that there is a pair of adjacent squares so that the numbers in them differ by at least 5.

Problem 8. On the straight line there is a finite colony of bacteries. Every moment bacteries which have neither left neighbour on the distance 1 nor right neighbour on the distance $\sqrt{2}$ will die. New bacteries are not born. Does there exist a colony which lives forever?

Problem 9. A plane is cut by several straight lines in a general position. Prove that for every straight line there is an adjacent triangle.

Problem 10. A six digit number is called *lucky* if 7 divides the sum of its digits. Are there any two consecutive lucky numbers?

Problem 11. Several identical coins are placed on the table without overlapping. Prove that there is a coin which touches no more than three other coins.

Problem 12. Several coins are placed on the table without overlapping. Prove that there is a coin which touches no more than five other coins.

Problem 13. Several coins are placed on the table without overlapping. Prove that one can slide one of the coins to the edge of the table without moving any other coin.

Problem 14. 30 numbers are placed on a circle such that every number is equal to the absolute value of the difference between next two numbers in the clockwise direction. Given that the sum of all numbers is 1 define all these numbers and how they are placed.

In the next problems one can apply *the method of small perturbations* which is related to *the method of extreme case*.

Problem 15. A straight line intersects a polygon exactly 2001 times. Prove that there is a straight line which is not parallel to any side of the polygon and has more than 2001 common points with it. **Problem 16.** There are 100 chords in a circle such that any two of them intersect. Is it allways possible to draw one more chord such that it intersects all of them?

3.3 Olympiads archive

Problem 1 (St. Petersburg Mathematical Olympiad). Several people are sitting at a round table. Each of of them has some nuts. At a certain signal, each person passes some of his nuts to the person sitting to his right: if he has an even number of nuts, he passes half of them. Otherwise, he takes one nut from the plate in the center and gives half to his neighbour on the right. Prove that at some moment everybody will have the same number of nuts.

Problem 2 (Tournament of Towns). Ten people are sitting at a round table. They have 100 nuts among them. At a certain signal, each person passes some of his nuts to the person sitting to his right: if he has an even number of nuts, he passes half of them. Otherwise, he passes one nut plus half of the remainder. Prove that at some moment everybody will have the same number of nuts.

Problem 3 (Tournament of Towns). A labyrinth" is an 8×8 chessboard with barriers between some pairs of neighboring squares. If a rook can traverse the entire board without crossing any barriers, the labyrinth is "good": otherwise, it is "bad". Are there more good labyrinths or more bad labyrinths?

Problem 4 (St. Petersburg Mathematical Olympiad) Positive integers are written in the cells of the rectangular table. One is allowed to double all the numbers of any row and also to substract 1 from every number of any column. Prove that applying these two operations many times one can get a table filled by zeroes.