Analysis of Typical Mistakes

Can It Happen or Is It True

Significant part of Tournament of the Towns problems deals with questions in one of the following forms:

- "Can it Happen?",
- "Is it possible?",
- "Does there exist?", or
- "Is it True?"

One should distinguish these two groups of questions. In order to give a positive answer on first group of questions it is sufficient to give an example (which shows that the situation, indeed, can happen). However, to show that the situation is not possible (can not happen, exist) example is not enough; proof (in general case) is needed. On the other hand, to answer positively on "Is it True" question, one must consider a general situation. However, to prove that the statement is not true one example (counterexample) is needed.

Problems

1. A hiker walked for 3.5 hours. For any one hour period he walked exactly 5 km.

   Is it true that his average speed was 5 km/h?

2. 2007 (TT Spring Round 2003 Juniors) dollars are placed into N purses, and the purses are placed into M pockets. It is known that N is greater than the number of dollars in any pocket. Is it true that there is a purse with less than M dollars in it?

3. (TT Spring Round 2001 Seniors) Does there exist a regular triangular prism that can be covered (without overlaping) by different equilateral triangles? (It is allowed to bent the triangles around the edges of the prism).

4. There are two polynomials in the form $P(x)=x^2+px+q$. Each root of the first polynomial is less than 100, while each root of the second polynomial is more than 100. Consider a polynomial that is equal to the sum of given polynomials. Is this possible that one of its roots is less than 100 while the second root is more than 100?

5. (Tournament of Towns) The intelligence quotient (IQ) of a country is defined as the average IQ of its entire population. It is assumed that the total population and individual IQ's remain constant throughout.

   (a) A group of people from country A has emigrated to country B. Show that it can happen that as a result, the IQ's of both countries have increased.

   (b) After this, a group of people from B, which may include immigrants from A, emigrates to A. Can it happen that the IQ's of both countries will increase again?

(c) A group of people from country A has emigrated to country B, and a group of people from B has emigrated to country C. It is known that as a result, the IQ's of all three countries have increased. After this, a group of people from C emigrates to B and a group of people from B emigrates to A. Can it happen that the IQ's of all three countries will increase again?

More problems to practice

1. (TT Spring Round 2006 Seniors.) Do there exist functions $p(x)$ and $q(x)$, such that $p(x)$ is an even function while $p(q(x))$ is an odd function (different from 0)?

2. (TT Fall Round 2006 Seniors.) Is it possible to split a prism into disjoint set of pyramids so that each pyramid has its base on one base of the prism, while its vertex on another base of the prism ?

3. (TT) An iceberg in the shape of a convex polyhedron is floating in the ocean. Could it happen that no less than 90% of its volume is below while no less than 50% of its surface is above sea level?

4. (TT Fall Round 2006 Juniors-Seniors.)

   When Ann meets new people, she tries to find out who is acquainted with who. In order to memorize it she draws a circle in which each person is depicted by a chord; moreover, chords corresponding to acquainted persons intersect (possibly at the ends), while the chords corresponding to non-acquainted persons do not. Ann believes that such set of chords exists for any company. Is her judgement correct?

5. (TT Fall Round 2005 Seniors.)

   A cube lies on the plane. After being rolled a few times (over its edges), it is brought back to its initial location with the same face up. Could the top face have been rotated by 90 degrees?

6. (TT Fall Round 2002 Seniors)

   a). A test was conducted in a class. It is known that at least 2/3 of the problems were hard: each such problem was not solved by at least 2/3 of the students. It is also known that at least 2/3 of students passed the test: each such student solved at least 2/3 of the suggested problems. Is this situation possible?

   b). The same question with 2/3 replaced by 3/4.

   c). The same question with with 2/3 replaced by 7/10.

7. Data (an android with the superior logic and calculating abilities) discovered an abandoned space station. The station made in a shape of torus is divided into identical cubicals. Each cubical is equipped with a tumbler that turns on or off light in it. Data's task is to figure out the number of cubicals on the station. However, the only thing he can do is to go along the station switching on or off lights. Can he fulfill his task (in finite time)? Initially, each tumbler could be either in OFF or ON position.

8. Is it possible to cut a circle into seven parts of equal area by three straight cuts?

Solutions

1. (TT Spring Round 2006 Seniors.) ANSWER:Yes. Consider functions $p(x) = \cos x$ and $q(x) = \pi/2 - x$. It is evident that $p(x)$ is an even function, while $p(q(x)) = \sin x$ is an odd function.

2. (TT Fall Round 2006 Seniors.) ANSWER: No. The total sum of volumes of the pyramids with bases on the bottom base of the prism does not exceed one third of the prism volume. The same is true for the pyramids with bases on the top base of the prism. Therefore, the total sum of volumes of all the pyramids is less than the volume of the prism. Contradiction.

3. (TT)

   ANSWER: Yes.

   Consider an upside-down regular pyramid with the 1×1 square base and height h. Then $9/10$ of h should be submerged, and the underwater part of surface is $2(9/10)^2 l$ while the total surface is $2l + 1$ where l is a height of a lateral face. So we need $2(9/10)^2 l \leq l + \frac{1}{2}$ or $62l < 50$. This is possible since we can take any $l > 0.5$.

4. (TT Fall Round 2006 Juniors-Seniors.)

   ANSWER: No.

   Counterexample. Consider a company: a host with three sons and three guests. The guests do not know each other, the host knows all the guests, while each son knows only two guests. No two sons know the same pair of the guests. It is clear, that the guests chords intersect the host chord in three distinct points; one point is between two others. So, the guest chord through this point separates two other guest chords. Therefore, the chord of the son who knows only two latter guests must intersect the guest chord in between. Contradiction.

5. (TT Fall Round 2005 Seniors.)

   ANSWER: No. The vertices of a cubic lattice may be painted black and white such that no two vertices of the same colour are adjacent. The vertices of the cube are painted in the same colours as the vertices of its initial position in the cubic lattice. When the cube is rolled over, its white vertices always go to white vertices of the cubic lattice, and its black vertices always go to black vertices of the cubic lattice. When it returns to its initial position, again with the letter A on its top face, the letter A cannot have made a 90 turn this requires the vertices of the cube to have different colours from the corresponding vertices of the cubic lattice

6. (TT Fall Round 2002 Seniors)

   Let N be the number of students in the class, M the number of the problems, P the number of passed students, H the number of hard problems. According to definition "a problem is hard" if it has not been solved by at least rN students; where $r = 2/3, 3/4, 7/10$ in (a), (b),(c). Also, according to definition "a student passes" if he solves at least rM problems.

   (a) It is possible. Example. Let S_1, S_2, S_3 be students, while P_1, P_2,P_3 be problems. Let S_1 solve P_1, P_3, S_2 solve P_2, P_3 while S_3 solve neither P_1 no P_2. Then, P_1 and P_2 are hard problems and S_1 and S_2 students who pass the test.

   (b) It is impossible. Let us write down the results of the test ("+" or "-") into $N \times M$ table. Let passed students be on the top and hard problems be on the left of the table. Let us estimate K_+ and K_- (the numbers of "+" and "$-$" in the table).

Then, $K_+ \geq$(number of "+" got by students who passed)$\geq P \times rM \geq r^2 \times M \times N$, and $K_- \geq$ (number of "$-$ " got for hard problems) $\geq H \times rN \geq r^2 \times M \times N$

Then $M \times N = (K_+ + K_-) \geq 2r^2 M \times N$. However, this is impossible for $r = 3/4$.

(c) It is impossible. Arguments of (b) do not work since in this case $2r^2 \leq 1$. Let us denote by K_+ and K_- the numbers of "+" and "$-$" in the top-left $P \times H$ sub-table.

Then $K_+ \geq$ (minimal number of "+ " for hard problems got by students who passed) $\geq P \times 4H/7$.

(A student cannot pass if he solves less than $4H/7$ of hard problems even if he solves all the easy problems, the number of which does not exceed $3M/7$).

On the other hand, $K_- \geq$ (minimal number of "$-$" got by students who passed for hard problems) $\geq H \times 4P/7$.

So, $P \times H = (K_+ + K_-) \geq 8P/7 \times H$. However, this is is impossible.

7. Answer: Yes. Assume that Data starts at cubical with light ON and travels in clockwise direction until he encounters a cubical with an ON light. He switches it to OFF. When he goes back to cubicle he started (counting). If in this cubicle the light is OFF then Data knows that he has circled the station; therefore, his task is fulfilled. Otherwise, Data goes clockwise direction again till he encounters a cubical with an ON light. Notice, that this time he travelled further distance from the original cubical. So, eventually, by repeating the procedure, Data fulfills his task.

8. Answer: No.